2.24 Example

Is the list $(1,2,3), (4,5,8), (9,6,7), (-3,2,8)$ linearly independent in $\mathbb{R}^3$? NO

The list $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{R}^3$. This list has length 3

- No list of length greater than 3 is lin. indep. in $\to \mathbb{R}^3$, b/c any vector in the list $(1,2,3), (4,5,8), (9,6,7), (-3,2,8)$ is a linear combo of the other three

2.25 Example Does the list $(1,2,3,-5), (4,5,8,3), (9,6,7,-1)$ span $\mathbb{R}^4$?

No, the list $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ spans

No list of length less than 4 spans $\mathbb{R}^4$.

2.26 Finite Dimensional Subspaces

Every subspace of a finite dimensional vector space is finite dimensional. (skipping the proof in lecture)

2.B Bases

2.27 Definition: A basis of V is a list of vectors $v_1, \ldots, v_m$ of V that is linearly independent & spans V.

2.28 Example

- The list $(1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,\ldots,0,1)$ is a basis of $\mathbb{F}^n$

  It is called the standard basis of $\mathbb{F}^n$
- $(1,0), (0,1)$ is a basis of $\mathbb{F}^2$
- $(2,3), (4,6)$ is NOT a basis of $\mathbb{F}^2$ (is linearly dependent)
- $(1,2,3), (2,1,3)$ is NOT a basis of $\mathbb{F}^3$ (only two vectors in $\mathbb{F}^3$, therefore does not span $\mathbb{F}^3$)
- $(1,2), (3,5), (4,1,3)$ is NOT a basis of $\mathbb{F}^2$

  (three vectors in $\mathbb{F}^2$, therefore one vector is linear combo of the other two) (lin. dependent).
- $(1,1,0), (0,0,1)$ is a basis of $\{(x,x,y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$ b/c $(x,x,y) = (x,x,0) + (0,0,y)$

  $= x(1,1,0) + y(0,0,1)$
- $(1,-1,0), (1,0,-1)$ is a basis of $\mathbb{F}\{(x,y,z) \in \mathbb{F}^3 : x+y+z=0\}$

  b/c $x+y+z=0$ implies $z=-x-y$.

$(x,y,z) = (x,y,-x-y)$

$= (x,0,-x) + (0,y,-y)$

$= x(1,0,-1) + y(0,1,-1)$

so $(1,0,-1), (0,1,-1)$ spans V.

Suppose $a, b \in \mathbb{F}$ satisfy

$a(1,0,-1) + b(0,1,-1) = (0,0,0)$

$(a,0,-a) + (0,b,-b) = (0,0,0)$

$(a,b,-a-b) = (0,0,0)$

Equate coordinates $a=0, b=0, -a-b=($

Therefore the list $(1,0,-1), (0,1,-1)$ is lin. indep. So

2.29 Criterion for Basis

A list $v_1, \ldots, v_n \in V$ is a basis of V if & only if every $v \in V$ can be written uniquely (in only one way) in the form $v = a_1v_1 + \ldots + a_nv_n$ for some $a_1, \ldots, a_n \in \mathbb{F}$

Proof forward direction:

Suppose $v_1, \ldots, v_n$ is a basis of V. For all $v \in V$, there exist $a_1, \ldots, a_n$ such that

$$* \quad v = a_1v_1 + \ldots + a_nv_n.$$

We need to show that this representation is unique

Suppose $c_1, \ldots, c_n \in \mathbb{F}$ also satisfy

$$\star \quad v = c_1v_1 + \ldots + c_nv_n$$

Subtracting $* - \star$, we get

$$0 = (a_1 - c_1)v_1 + \ldots + (a_n - c_n)v_n$$

Since 0 can be written in only one way, we must have

$$a_1 - c_1 = 0, \ldots, a_n - c_n = 0$$

Therefore,

$a_1 = c_1, \ldots, a_n = c_n$ and so the representation is uniqu

Backward direction

Suppose we can write every $v \in V$ uniquely into form

$$v = a_1v_1 + \ldots + a_nv_n$$

Then v is a linear combination of $v_1, \ldots, v_n$, which means $v_1, \ldots, v_n$ spans V.

Now we must prove that $v_1, \ldots, v_n$ is linearly independent.

Suppose $a_1, \ldots, a_n \in \mathbb{F}$ satisfy $a_1v_1 + \ldots + a_nv_n = 0$

Since the representation of every $v \in V$ ($v = 0$, in particular) is unique, we must have $a_1 = 0, \ldots, a_n = 0$

Therefore, $v_1, \ldots, v_n$ is lin independent

Since we proved that $v_1, \ldots, v_n$ is a lin indep. set that spans V, we conclu that $v_1, \ldots, v_n$ is a basis of V.

2.31 Spanning list contains a basis

Every spanning list in a vector space can be reduced to a basis of the vector space

For example, $(1,0), (0,1), (2,3)$ is a spanning list of $\mathbb{F}^2$. But we can reduce this list to $(1,0), (0,1)$

basis of $\mathbb{F}^2$

also a basis of $\mathbb{F}^2$

Proof skipped for 2.31

2.28 cont.

The list $1, z, \ldots, z^m$ is a basis of $P_m(\mathbb{F})$

every polynomial $p \in P_m(\mathbb{F})$ is written $p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_m z^m$ for all $z \in \mathbb{F}$, for all $a_0, a_1, a_2, \ldots, a_m \in \mathbb{F}$. Notice that $p(z)$ is a linear combo of $1, z, z^2, \ldots, z^m$

Also for example

$$1, z+3, (z+3)^2, \ldots, (z+3)^n \text{ is a basis of } P_m(\mathbb{F})$$

Another Example, $1, z-6, (z-6)^2$ is a basis of $P_2(\mathbb{F})$

2.32 Basis of finite dimensional vector space

Every finite-dimensional vector space has a basis

Proof: According to 2.10 of Axler there exist a spanning list of every finite-dimensional vector space. By 2.31, the spanning list can be reduced to a basis

2.33 Linearly Independent list extends to a basis

Let V be a finite-dimensional vector space, and let $u_1, \ldots, u_m \in V$ be a linearly indep. list. Then this list can be extended to a basis $w_1, \ldots, w_n \in V$.

Proof: Suppose $u_1, \ldots, u_m \in V$ is lin. indep. & $w_1, \ldots, w_n \in V$ is a bo
Then the list

$u_1, \ldots, u_m, w_1, \ldots, w_n$

spans V, Apply the procedure of 2.31 of Axler to remove some vectors from $u_1, \ldots, u_m, w_1, \ldots, w_n$ (if necessary) to reduce this spanning list to a basis of V

For Example, the list $(2,3,4), (9,6,8)$ is lin indep. in $\mathbb{F}^3$
Then following the procedure of the proof of 2.33 of Axler, we can obtain a list $(2,3,4), (9,6,8), (0,1,0)$, which is a basis of $\mathbb{F}^3$.

2.34 Every subspace of V is part of a direct sum equal to V.

Suppose V is a finite-dimensional vector space, & U is a subspac of V. Then there exists a subspace W of V such that $V = U \oplus W$
can write $v = u + w$ in only one way $v \in V$, $u \in U$, $w \in W$

Proof: Since V is finite dimensional & U is a subspace of V, by 2.26 of Axler, U is also finite dimensional. By 2.32 of Axle there exists a basis $u_1, \ldots, u_m \in U$. This means in particular that $u_1, \ldots, u_m$ is linearly indep. in V. By 2.33 of Axler, we can extend $u_1, \ldots, u_m$ to a basis $u_1, \ldots, u_m, w_1, \ldots, w_n$ of V.
Now, let $W = \text{span}(w_1, \ldots, w_n)$

To prove $V = U \oplus W$. By 1.45 of Axler, we just need to prove

$V = U + W$ & $U \cap W = \{0\}$

First, we will prove $V = U + W$

Suppose $v \in V$. Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ spans V, there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$ such that $v = \underbrace{a_1 u_1 + \ldots + a_m u_m}_{u} + \underbrace{b_1 w_1 + \ldots + b_n w_n}_{w}$

so $v = u + w$, where $u \in U$ & $w \in W$. so $V = U + W$

Proof: Next, we will show $U \cap W = \{0\}$

Suppose $v \in U \cap W$. Then there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in F$ such that $v = a_1u_1 + \ldots + a_mu_m = b_1w_1 + \ldots + b_nw_n$

Subtracting we get

$$a_1v_1 + \ldots + a_mv_m - b_1w_1 - \ldots - b_nw_n = 0$$

Since $v_1, \ldots, v_m, w_1, \ldots, w_n$ are linearly indep., all the scalars are zero

$$a_1 = 0, \ldots, a_m = 0, -b_1 = 0, \ldots, -b_n = 0.$$

$$(b_1 = 0, \ldots, b_n = 0).$$

Therefore, $v = a_1v_1 + \ldots + a_mv_m$
$= 0u_1 + \ldots + 0u_m$
$= 0 \in \{0\}$

and $v = b_1w_1 + \ldots + b_nw_n$
$= 0w_1 + \ldots + 0w_n$
$= 0 \in \{0\}$

Therefore, $U \cap W \subset \{0\}$

so $U \cap W = \{0\}$, as desired

2.C Dimension

We need to recall 2.23 of Axler section 2.B

2.23: Length of lin indep. list $\leq$ length of spanning list in a finite-dimensional vector space

2.35 The length of a basis of a vector space does not depend on the basis

Any two bases of a finite dimensional vector space have the same length (same number of vectors in the bases)

Proof: Suppose V is a finite dim vector space. Let $B_1 = v_1, \ldots, v_n$ and $B_2 = w_1, \ldots, w_n$ be two bases of V. Then, by 2.23 of Axler, the length of B_1 is less than or equal to the length of B_2.

$B_1 = v_1, \ldots, v_m$
$B_2 = w_1, \ldots, w_n$
$v_1, \ldots, v_m, w_1, \ldots, w_n$

Interchange (swap) the roles of B_1 & B_2. By 2.23 of Axler, the length of B_2 is $\leq$ the length of B_1.

In other words, we have length of $B_1 \leq$ length of B_2 and length of $B_2 \leq$ length of B_1

Therefore, length of B_1 = length of B_2.

In other words, the two bases have the same length.