LECTURE 06
07-02-19

2.A Let's Revisit

2.7 Span is the smallest containing subspace

Every subspace of V containing $v_1, \dots, v_m$ (each v_j, $j=1,\dots,m$) contains $\text{span}(v_1, \dots, v_m)$

In other words:

If a subspace contains $v_1, \dots, v_m$, then it contains $\text{span}(v_1, \dots, v_m)$

Proof of that statement:

Let U be a subspace of V that contains $v_1, \dots, v_m$. Since subspaces are closed under addition and scalar multiplication, $v_1, \dots, v_m \in U$ implies

$$a_1v_1 + \cdots + a_mv_m \in U$$

for all $a_1, \dots, a_m \in \mathbb{F}$. By definition,

$$\text{span}(v_1, \dots, v_m) = \{a_1v_1 + \cdots + a_m : a_1, \dots, a_m \in \mathbb{F}\}$$

Therefore, $\text{span}(v_1, \dots, v_m) \subset U$

In other words, every subspace of V that contains $v_1, \dots, v_m \in V$ must also contain $\text{span}(v_1, \dots, v_m)$. This makes $\text{span}(v_1, \dots, v_m)$ the SMALLEST subspace of V that contains $v_1, \dots, v_m$.

2.21 Linear Dependence Lemma

Suppose $v_1, \dots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \dots, m\}$ such that:

(a) $v_j \in \text{span}(v_1, \dots, v_{j-1})$

($v_j = a_1v_1 + \cdots + a_{j-1}v_{j-1}$ for some $a_1, \dots a_{j-1} \in \mathbb{F}$)

(b) if the j^{th} term is removed from $v_1, \dots, v_m$ (resulting in $v_1, \dots v_{j-1}, v_{j+1}, \dots v_m$)

then $\text{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m) = \text{span}(v_1, \dots, v_m)$

- Proof of (a):

Since $v_1, \dots, v_m$ is linearly dependent, there exist $a_1, \dots, a_m \in \mathbb{F}$, not all 0, such that

$a_1 v_1 + \cdots + a_m v_m = 0$

In particular, there exists $j \in \{1, \dots, m\}$ such that $a_j \neq 0$.

So we have

$$a_1 v_1 + \cdots + a_{j-1} v_{j-1} + a_j v_j + \cdots + a_m v_m = 0$$

Solve for v_j:

$$v_j = -\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}$$

In other words v_j is a linear combination of $v_1, \dots, v_{j-1}$

Therefore, $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$. proving (a)

Proof of (b):

Suppose $u \in \operatorname{span}(v_1, \dots, v_m)$. Then there exist $c_1, \dots, c_m \in \mathbb{F}$ such that

$$u = c_1 v_1 + \cdots + c_m v_m$$

Recall from the proof of part (a):

$$v_j = -\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}$$

we have

$$\begin{aligned}
u &= c_1 v_1 + \cdots + c_m v_m \\
&= c_1 v_1 + \cdots + c_{j-1} v_{j-1} + c_j v_j + c_{j+1} v_{j+1} + \cdots + c_m v_m \\
&= \boxed{c_1 v_1} + \cdots + \boxed{c_{j-1} v_{j-1}} + c_j \left(-\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1} \right) + c_{j+1} v_{j+1} + \cdots + c_m v_m \\
&= \left(c_1 - c_j \frac{a_1}{a_j} \right) v_1 + \cdots + \left(c_{j-1} - c_j \frac{a_{j-1}}{a_j} \right) v_{j-1} + c_{j+1} v_{j+1} + \cdots c_m v_m \\
&\in \operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m)
\end{aligned}$$

Therefore, $\operatorname{span}(v_1, \dots, v_m) \subset \operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m)$

On the other hand, let $u \in \operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m)$

Then there exist $a_1, \dots, a_m \in \mathbb{F}$ such that

$$u = a_1 v_1 + \cdots a_{j-1} v_{j-1} + a_{j+1} v_{j+1} + \cdots + a_m v_m$$

But also,

$$\begin{aligned}
u &= a_1 v_1 + \cdots + a_{j-1} v_{j-1} + \boxed{0 v_j} + a_{j+1} v_{j+1} + \cdots a_m v_m \\
&\in \operatorname{span}(v_1, \dots, v_m)
\end{aligned}$$

So $\operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m) \subset \operatorname{span}(v_1, \dots, v_m)$

Therefore, we have the set equality $\operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m) = \operatorname{span}(v_1, \dots, v_m)$ proving part (b)

2.24 Example Is the list $(1,2,3),(4,5,8),(9,6,7),(-3,2,8)$ linearly independent in $\mathbb{R}^3$?

The list $(1,0,0),(0,1,0),(0,0,1)$ spans $\mathbb{R}^3$

This list has length 3.

No list of length greater than 3 is linearly independent in $\mathbb{R}^3$

because any vector in the list

$(1,2,3),(4,5,8),(9,6,7),(-3,2,8)$

is a linear combination of the other three

2.25 Example Does the list $(1,2,3,-5),(4,5,8,3),(9,6,7,-1)$ span $\mathbb{R}^4$?

No. The list $(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$ span $\mathbb{R}^4$

No list of length less than 4 spans $\mathbb{R}^4$

2.26 Finite dimensional subspaces

Every subspace of a finite-dimensional vector space is finite dimensional.

We skipping the proof in our lecture

2.B Bases

2.27 Definition

A basis of V is a list of vectors $v_1, \dots, v_m$ of V that is linearly independent and spans V.

2.28 Example

- The list $(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)$ is a basis of $\mathbb{F}^n$
  It is called the standard basis of $\mathbb{F}^n$
- $(1,0),(0,1)$ is a basis of $\mathbb{F}^2$
  (standard basis)
- $(1,2),(3,5)$ is a basis of $\mathbb{F}^2$
- $(2,3),(4,6)$ is NOT a basis of $\mathbb{F}^2$
  (It is linearly dependent)
- $(1,2,3),(2,1,3)$ is NOT a basis of $\mathbb{F}^3$
  (only two vectors in $\mathbb{F}^3$, therefore does not span $\mathbb{F}^3$)

- $(1,2), (3,5), (4,13)$ is NOT a basis of $\mathbb{F}^2$
  (three vectors in $\mathbb{F}^2$, therefore one vector is linear combination of the other two)
  (linearly dependent)
- $(1,1,0), (0,0,1)$ is a basis of $\{(x,x,y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$
  because $(x,x,y) = (x,x,0) + (0,0,y)$
  $= x(1,1,0) + y(0,0,1)$
- $(1,-1,0), (1,0,-1)$ is a basis of $\{(x,y,z) \in \mathbb{F}^3 : x+y+z=0\}$
  because $x+y+z=0$ implies $z = -x-y$. So

2.29 Criterion for basis

A list $v_1, \dots, v_n \in V$ is a basis of V if and only if every $v \in V$ can be written uniquely (in only one way) in the form

$$v = a_1 v_1 + \cdots a_n v_n$$

for some $a_1, \dots, a_n \in \mathbb{F}$

Proof: Forward direction

Suppose $v_1, \dots, v_n$ is a basis of V. For all $v \in V$, there exist $a_1, \dots, a_n$ such that

$$v = a_1 v_1 + \cdots + a_n v_n \quad \star$$

we need to show that this representation is unique.

Suppose $c_1, \dots, c_n \in \mathbb{F}$ also satisfy

$$v = c_1 v_1 + \cdots + c_n v_n \quad \star$$

Subtract $\star - \star$, we get

$$0 = (a_1 - c_1) v_1 + \cdots + (a_n - c_n) v_n$$

Since 0 can be written in only one way, we must have

$$a_1 - c_1 = 0, \dots, a_n - c_n = 0$$

Therefore,

$$a_1 = c_1, \dots, a_n = c_n$$

and so the representation is unique

Backward direction

Suppose we can write every $v \in V$ uniquely into form

$$v = a_1 v_1 + \cdots + a_n v_n$$

Then v is a linear combination of $v_1, \dots, v_n$, which means $\boxed{v_1, \dots, v_n \text{ spans } V.}$

Now we must prove that $v_1, \dots, v_n$ is linearly independent

Suppose $a_1, \dots, a_n \in \mathbb{F}$ satisfy

$$a_1 v_1 + \cdots + a_n v_n = 0 \quad v = 0$$

Since the representation of every $v \in V$ ($v = 0$, in particular) is <u>unique</u>, we mush have

$$a_1 = 0, \dots, a_n = 0$$

Therefore $\boxed{v_1, \dots, v_n \text{ is linearly independent.}}$

since we proved that $v_1, \dots, v_n$ is a linearly independent set that spans V, we conclude that $v_1, \dots, v_n$ in a <u>basis</u> of V.

2.31 Spanning list contains a basis

Every spanning list in a vector space can be reduced to a basis of the vector space.

<u>For example,</u> basis of $\mathbb{F}^2$ basis of $\mathbb{R}^2$

$\boxed{(1,0)}\ \boxed{(0,1)}$, $\boxed{(2,3)}$ is a spanning list of $\mathbb{F}^2$ But we can reduce this list to

$$(1,0)\ (0,1)$$

Proof skipped for 2.31

2.32 Basis of finite dimensional vector space

Every finite-dimensional vector space has a basis.

Proof: According to 2.10 for Axler (definition of finite dimensional vector space), there exists a spanning list of every finite-dimensional vector space. By 2.31 the spanning list can be ~~reduced~~ reduced to a basis

2.33 Linearly independent list extends to a basis

~~Every~~ let V be a finite-dimensional vector space, and let $v_1, \dots, v_m \in V$ be a linearly independent list. Then this list can be extended to a basis $w_1, \dots, w_n$ of V.

Proof: Suppose $u_1, \dots, u_m \in V$ is linearly independent and $w_1, \dots, w_n \in V$ is a basis. Then the list

$$u_1, \dots, u_m, w_1, \dots, w_n$$

extend to a spanning list of V

spans V. Apply the procedure 2.31 of Axler to remove some vectors from $u_1, \dots, u_m, w_1, \dots, w_n$ (if necessary) to reduce this spanning list to a basis of V.

For example, the list $(2,3,4), (9,6,8)$ is linearly independent in $\mathbb{F}^3$. Then following the procedure of the proof of 2.33 of Axler, we can obtain a list

$$(2,3,4), (9,6,8), (0,1,0)$$

which is a basis of $\mathbb{F}^3$

2.34 Every subspace of V is part of a direct sum equal to V

Suppose V is finite dimensional vector space, and U is a subspace of V. Then there exists a subspace W of V such that $V = U \oplus W$

can write $v = u + w$ in only one way $v \in V$, $u \in U$, $w \in W$

Proof: Since V is finite dimensional and U is a subspace of V, by 2.26 of Axler, U is also finite-dimensional. By 2.32 of Axler, there exists a basis $u_1, \dots, u_m$ of U. This means in particular that $u_1, \dots, u_m$ is linearly independent in V. By 2.33 of Axler, we can extend $u_1, \dots, u_m$ to a basis $u_1, \dots, u_m, w_1, \dots, w_n$ of V. Now, let $W = \text{span}(w_1, \dots, w_n)$

To prove $V = U \oplus W$. By 1.45 of Axler, we just need to prove

$$V = U + W \text{ and } U \cap W = \{0\}$$

First, we will prove $V = U + W$

Suppose $v \in V$. Since $u_1, \ldots, u_m, w_1, \ldots w_n$ spans V, there exist $a_1, \ldots, a_m, b_1, \ldots b_n \in \mathbb{F}$ such that

$$v = \underbrace{\boxed{a_1 v_1 + \cdots a_m v_m}}_{u} + \underbrace{\boxed{b_1 w_1 + \cdots + b_n w_n}}_{w}$$

so $v = v + w$, where $u \in U$ and $w \in W$

Next, we will show $U \cap W = \{0\}$ $\quad v \in U$ & $u \in W$

Suppose $v \in U \cap W$. Then there exists $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$ such that

$$v = a_1 u_1 + \cdots + a_m u_m = b_1 w_1 + \cdots + b_n w_n$$

Subtracting we get

$$a_1 u_1 + \cdots + a_n u_n - b_1 w_1 - \cdots - b_n w_n = 0$$

Since $u_1, \ldots, u_m, w_1, \ldots w_n$ are linearly independent, all the scalars are zero:

$$a_1 = 0, \ldots, a_m = 0, -b_1 = 0, \ldots, -b_n = 0$$
$$(b_1 = 0, \ldots, b_n = 0)$$

Therefore,

$$\begin{aligned} v &= a_1 v_1 + \cdots + a_m u_m \\ &= 0u_1 + \cdots + 0u_m \\ &= 0 \in \{0\} \end{aligned}$$

and

$$\begin{aligned} v &= b_1 w_1 + \cdots + b_n w_n \\ &= 0w_1 + \cdots + 0w_n \\ &= 0 \in \{0\} \end{aligned}$$

Therefore, $U \cap W \subset \{0\}$

so $U \cap W = \{0\}$, as desired