**2.24 Example** Is the list $(1,2,3), (4,5,8), (9,6,7), (-3,2,8)$ linearly independent in $\mathbb{R}^3$?. No.

The list $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{R}^3$. This list has length 3.

- No list of length greater than 3 ~~spans $\mathbb{R}^3$~~ is linearly independent in $\mathbb{R}^3$, because any vector in the list $(1,2,3), (4,5,8), (9,6,7), (-3,2,8)$ is a linear combination of the other three.

**2.25 Example** Does the list $(1,2,3,-5), (4,5,8,3), (9,6,7,-1)$ span $\mathbb{R}^4$?

No. the list $(1,0,0,0), (0,1,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ spans $\mathbb{R}^4$.

No list of length less than 4 spans $\mathbb{R}^4$.

**2.26 Finite dimensional subspaces**

Every subspace of a finite-dimensional vector space is finite dimensional.

We're skipping the proof in our lecture.

## 2.B Bases

**2.27 Definition**

A <u>basis</u> of $V$ is a list of ~~$\notin$~~ vectors $v_1, \dots, v_m$ of $V$ that is linearly independent and spans $V$.

**2.28 Example**

- The list $(1,0,\dots,0), (0,1,0\dots,0), \dots, (0,\dots,0,1)$ is a basis of $\mathbb{F}^n$. It is called the <u>standard basis</u> of $\mathbb{F}^n$.
- $(1,0), (0,1)$ is a basis of $\mathbb{F}^2$. (standard basis)
- $(1,2), (3,5)$ is a basis of $\mathbb{F}^2$.
- $(2,3), (4,6)$ is NOT a basis of $\mathbb{F}^2$. (It is linearly <u>dependent</u>)

- $(1,2,3), (2,1,3)$ is NOT a basis of $\mathbb{F}^3$.
  (only two vectors in $\mathbb{F}^3$, therefore does not span $\mathbb{F}^3$).

- $(1,2), (3,5), (4,13)$ is NOT a basis of $\mathbb{F}^2$.
  (three vectors in $\mathbb{F}^2$, therefore one vector is linear combination of the other two) (linearly dependent)

- $(1,1,0), (0,0,1)$ is a basis of $\{(x,x,y) \in \mathbb{F}^3 : x,y \in \mathbb{F}\}$.
  because $(x,x,y) = (x,x,0) + (0,0,y) = x(1,1,0) + y(0,0,1)$

- $(1,-1,0), (1,0,-1)$ is a basis of $U = \{(x,y,z) \in \mathbb{F}^3 : x+y+z=0\}$, because $x+y+z=0$ implies $z=-x-y$. So

$$
\begin{aligned}
(x,y,z) &= (x,y,-x-y) \\
&= (x,0,-x) + (0,y,-y) \\
&= \cancel{(x,0,-x)+(0,y,0)} + \cancel{(x,0,0} \\
&= x(1,0,-1) + y(0,1,-1)
\end{aligned}
$$

So $(1,0,-1), (0,1,-1)$ spans $U$.

Suppose $a,b \in \mathbb{F}$ satisfy $a(1,0,-1) + b(0,1,-1) = (0,0,0)$

$(a,0,-a) + (0,b,-b) = (0,0,0)$

$(a,b,-a,-b) = (0,0,0)$

Equate coordinates $\boxed{\begin{matrix} a=0 \\ b=0 \end{matrix}}$

$-a-b=0$

Therefore, the list $(1,0,-1),(0,1,-1)$ is linearly independent. So $(1,0,-1),(0,1,-1)$ is a basis of $U$.

## 2.29 Criterion for basis

A list $v_1, \dots, v_n \in V$ is a basis of $V$ if and only if every $v \in V$ can be written uniquely (in only one way) in the form

$$v = a_1 v_1 + \dots + a_n v_n \quad \text{for some } a_1, \dots, a_n \in \mathbb{F}.$$

### Proof

#### Forward direction

Suppose $v_1, \dots, v_n$ is a basis of $V$. For all $v \in V$, there exist $a_1, \dots, a_n$ such that ☆ $v = a_1 v_1 + \dots + a_n v_n$

We need to show that this representation is unique.

Suppose $c_1, \dots, c_n \in \mathbb{F}$ also satisfy ☆ $v = c_1 v_1 + \dots + c_n v_n$.

Subtracting ☆ − ☆, we get $0 = (a_1 - c_1)v_1 + \dots + (a_n - c_n)v_n$.

Since $0$ can be written in only one way, we must have

$a_1 - c_1 = 0, \; \dots, \; a_n - c_n = 0$.

Therefore, $a_1 = c_1, \dots, a_n = c_n$. and also the representation is unique.

## Backward direction

Suppose we can write every $v \in V$ uniquely in the form

$$v = a_1v_1 + \dots + a_nv_n.$$

Then $v$ is a linear combination of $v_1, \dots, v_n$, which means $v_1, \dots, v_n$ spans $V$.

Now we must prove that $v_1, \dots, v_n$ is linearly independent. Suppose $a_1, \dots, a_n \in \mathbb{F}$ satisfy

$$a_1v_1 + \dots + a_nv_n = 0 \quad v=0$$

Since the representation of every $v \in V$ ($v = 0$, in particular) is unique, we must have $a_1 = 0, \dots, a_n = 0$.

Therefore, $\boxed{v_1, \dots, v_n \text{ is linearly independent.}}$

Since we prove that $v_1, \dots, v_n$ is a linearly independent set that spans $V$, we conclude that $v_1, \dots, v_n$ is a basis of $V$.

## 2.31 Spanning list contains a basis

Every spanning list in a vector space can be reduced to a basis of the vector space.

basis of $\mathbb{F}^2$    basis of $\mathbb{F}^2$    Proof skipped for 2.31.

For example, $(1,0), (0,1), (2,3)$.

is a spanning list of $\mathbb{F}^2$. But we can reduce this list to $(1,0), (0,1)$.

## 2.32 Basis of finite-dimensional vector space

Every finite-dimensional vector space has a basis.

Proof: According to 2.10 of Axler (definition of finite-dimensional vector space), there exists a spanning list of every finite-dimensional vector space. By 2.31, the spanning list can be reduced to a basis.

## 2.33 Linearly independent list extends to a basis

Let $V$ be a finite-dimensional vector space, and let $u_1, \dots, u_m \in V$ be a linearly independent list. Then this list can be extended to a basis $w_1, \dots, w_n$ of $V$.

extend to a spanning list of $V$

Proof: Suppose $u_1, \dots, u_m \in V$ is linearly independent and $w_1, \dots, w_n \in V$ is a basis. Then the list $u_1, \dots, u_m, w_1, \dots, w_n$ spans $V$.
Apply the procedure of 2.31 of Axler to remove some vectors from $u_1, \dots, u_m, w_1, \dots, w_n$ (if necessary) to reduce this spanning list to a basis of $V$.

For example, the list $(2,3,4), (9,6,8)$ is linearly independent of $\mathbb{F}^3$.
Then following the procedure of the proof of 2.33 of Axler,
we can obtain a list $(2,3,4), (9,6,8), (0,1,0)$ which is a basis of $\mathbb{F}^3$.

## 2.34 Every subspace of V is part of a direct sum equal to V

Suppose $V$ is a finite-dimensional vector space, and $U$ is a subspace of $V$. Then there exists a subspace $W$ of $V$ such that $V = U \oplus W$.

Can write $v = u + w$ in only one way
$v \in V$, $u \in U$, $w \in W$.

Proof: Since $V$ is finite dimensional and $U$ is a subspace of $V$, by 2.26 of Axler, $U$ is also finite-dimensional. By 2.32 of Axler, there exists a basis $u_1, \dots, u_m$ of $U$. This means in particular that $u_1, \dots, u_m$ is linearly independent in $V$. By 2.33 of Axler, we can extend $u_1, \dots, u_m$ to a basis $u_1, \dots, u_m, w_1, \dots, w_n$ of $V$.

Now, let $W = \operatorname{span}(w_1, \dots, w_n)$.

To prove $V = U \oplus W$. By 1.45 of Axler, we just need to prove

$$V = U + W \text{ and } U \cap W = \{0\}.$$

First, we will prove $V = U + W$.

Suppose $v \in V$. Since $u_1, \dots, u_m, w_1, \dots, w_n$ spans $V$, there exist ~~$a_1, \dots, a_m$~~ $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ such that

$$v = \underbrace{\boxed{a_1u_1 + \cdots + a_m v_m}}_{u} + \underbrace{\boxed{b_1w_1 + \cdots + b_n w_n}}_{w}.$$

So $v = u + w$, where $u \in U$ and $w \in W$. So $V = U + W$.

Proof: Next, we will show $U \cap W = \{0\}$.

Suppose $v \in U \cap W$. Then $u \in U$ and $u \in W$. So there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$ such that $v = a_1u_1 + \cdots + a_m u_m = b_1w_1 + \cdots + b_n w_n$.

Subtracting, we get $a_1u_1 + \cdots + a_m u_m - b_1w_1 - \cdots - b_n w_n = 0$.

Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ are linearly independent, all the scalars are zero: $a_1 = 0, \ldots, a_m = 0, \ -b_1 = 0, \ldots, -b_n = 0$.

$(b_1 = 0, \ldots, b_n = 0)$

Therefore,
$$\begin{aligned} v &= a_1u_1 + \cdots + a_m u_m \\ &= 0u_1 + \cdots + 0u_m \\ &= 0 \in \{0\}. \end{aligned}$$

and
$$\begin{aligned} v &= b_1w_1 + \cdots + b_n w_n \\ &= 0w_1 + \cdots + 0w_n \\ &= 0 \in \{0\}. \end{aligned}$$

Therefore, $U \cap W \subset \{0\}$

So $U \cap W = \{0\}$, as desired.

---

Add to 2.28

- The list $1, z, \ldots, z^m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

  Every polynomial $p \in \mathcal{P}_m(\mathbb{F})$ is written $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$ for all $\boxed{z \in \mathbb{F}}$, for all $a_0, a_1, a_2, \ldots, a_m \in \mathbb{F}$

  Notice that $p(z)$ is a linear combination for $1, z, z^2, \ldots, z^m$.

Also, for example, $1, z+3, (z+3)^2, \ldots, (z+3)^m$, is a basis of $\mathcal{P}_m(\mathbb{F})$.

Another example, $1, z-6, (z-6)^2$ is a basis of $\mathcal{P}_2(\mathbb{F})$.