17/3 2.28 Example revisit

list $(1,-1,0),(1,0,-1)$ is a basis of $\{(x,y,z)\in\mathbb{F}^3: x+y+z=0\}$

$(x,y,z)=(x,y,-x-y)$

$=(x,0,-x)+(0,y,-y)$

$=x(1,0,-1)+y(0,1,-1)$

So $(1,0,-1),(0,1,-1)$ spans V.

Now check l.I.

let $a,b\in\mathbb{F}$

$a(1,0,-1)+b(0,1,-1)=(0,0,0)$

$(a,0,-a)+(0,b,-b)=(0,0,0)$

$(a,b,-a-b)=(0,0,0)$

$a=0$ $b=0$ $-a-b=0$

The list is l.I and a basis of V.

2.C Dimension

Recall 2.23 from section 2.B.
2.23: length of LI list is $\leq$ length of spanning list in a finite-dim. vector space

2.35 The length of a basis of a vector space does not depend on the basis
- Any 2 basis of a finite-dim. vector space have the same length (same # of vectors).
- Proof: Suppose V is a finite-dim. vector space. Let $B_1 = v_1, \ldots, v_m$ and $B_2 = w_1, \ldots, w_n$ be 2 bases of V. Then by 2.23 of length B_1 is less than or = to the length of B_2. $B_1 \leq B_2$
Interchange B_1 & B_2.
length of $B_2 \leq B_1$
So, length of $B_1 \leq B_2$ & $B_2 \leq B_1$
therefore $B_1 = B_2$ in length $m = n$ ✓

2.36 Def
Dimension of a finite-Dim vector space V is the length of any bases B of V.
The Dimension of $V = \dim V$

2.37 Examples
- $\dim \mathbb{F}^n = n$ because the length of any basis of $\mathbb{F}^n$ is n
- $\dim \mathcal{P}_m(\mathbb{F}) = m+1$ because for examples $1, z, z^2, \ldots, z^m$ is a basis of $\mathcal{P}_m(\mathbb{F})$ and length of the basis is $m+1$

2.38 Dimension of a subspace

If V is finite-dim and U is a subspace of V, then $\dim U \leq \dim V$.

Proof: Since V is finite-dim & U is a subspace of V, by 2.26, U is also finite-dim. By 2.32, $\exists$ a basis of U & a basis of V.

So the basis of U is LI list in V, & the basis of V is a spanning list of V.

The length of the LI list in U is $\dim U$. Also, the length of our spanning list in V is $\dim V$.

So $\dim U \leq \dim V$ ∎

2.39 LI list of the right length is a basis

Suppose V is finite-dim. Then every LI list of vectors in V with length $\dim V$ is a basis of V.

Proof: Suppose $\dim V = n$ let $v_1, \ldots, v_n$ be a LI list in V. By 2.33, extend $v_1, \ldots, v_n$ to a basis of V. Every basis of V has length n. Since $v_1, \ldots, v_n$ already has length n, we do not extend. This means $v_1, \ldots, v_n$ is itself a basis of V. ∎

2.40 Example

Show that list $(5,7), (4,3)$ is a basis of $\mathbb{F}^2$.

Proof: Suppose $a_1, a_2 \in \mathbb{F}$

$$a_1(5,7) + a_2(4,3) = 0$$

$$5a_1, 7a_1 + 4a_2, 3a_2 = (0,0)$$

$$5a + 4a_2, 7a + 3a_2 = (0,0)$$

$$5a_1 + 4a_2 = 0$$

$$7a_1 + 3a_2 = 0 \quad a_1 = 0 \; a_2 = 0$$

The list $(5,7), (4,3)$ is LI in $\mathbb{F}^2$. Since length is 2 it is a basis.

2.41 Example

Show that $1, (x-5)^2, (x-5)^3$ is a basis of the subspace U of $P_3(\mathbb{R})$ defined by

$$U = \{p \in P_3(\mathbb{R}) : p'(5) = 0\}$$

let $p_1(x) = 1$ $p_1'(x) = 0$ so $p_1'(5) = 0$ $p_1 = 1 \in U$
$p_2(x) = (x-5)^2$ $p_2'(x) = 2(x-5)$ so $p_2'(5) = 0$ $p_2 = (x-5)^2 \in U$
$p_3(x) = (x-5)^3$ $p_3'(x) = 3(x-5)^2$ so $p_3'(5) = 0$ $p_3 = (x-5)^3 \in U$
So $1, (x-5)^2, (x-5)^3 \in U$

Suppose $a, b, c \in \mathbb{R}$

$$a + b(x-5)^2 + c(x-5)^3 = 0 \quad \forall x \in \mathbb{R}$$

$c = 0$ $b = 0$ implies $a = 0$. So $1, (x-5)^2, (x-5)^3$ is lI
Since length is 3 and $\dim U$ is 3, by 2.39 the list is a basis

2.42 Spanning list of the right length is a basis

Suppose V is finite-dim. vector space. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

Proof: Suppose $\dim V = n$ & $v_1, \ldots, v_n$ spans V. By 2.31, reduce if needed to a basis of V. But length is n so don't reduce. Therefore, $v_1, \ldots, v_n$ itself is a basis of V.

2.43 Dimension of a sum

If U_1 & U_2 are subspaces of a finite-dim vector space V, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Proof: Let $m = \dim(U_1 \cap U_2)$, & let $u_1, \ldots, u_m$ be a basis of $U_1 \cap U_2$ then U_1 is lI.
By 2.33, extend $u_1, \ldots, u_m, v_1, \ldots, v_j$ of U_1 which means $\dim U_1 = m + j$.

$u_1, \dots, u_m$ is ℓI in V_2

extend: $u_1, \dots, u_m, w_1, \dots, w_k$ of V_2 so $\dim V_2 = m+k$

$V_1, V_2 \subset \operatorname{span}(u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k)$ means

$= V_1 + V_2$

Why does $V = \{p \in P_3(\mathbb{R}) : p'(5) = 0\}$ have $\dim = 3$?

Since $1, (x-5)^2, (x-5)^3$ is ℓI in V, $\dim V$ is 3 or 4.

Since V is a subspace of $P_3(\mathbb{R})$

$3 \leq \dim V \leq \dim P_3(\mathbb{R}) = 4$

$x-5 \in P_3(\mathbb{R})$, but $q = x-5 \notin V$ $\quad q'(x) = 1$

$q'(5) = 1$

$q'(5) \neq 0$

So $q \notin V$

So $V \neq P_3(\mathbb{R})$

So $3 \leq \dim V < \dim P_3(\mathbb{R}) = 4$

so $\dim V = 3$

Prove $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is ℓI

Suppose $a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_j v_j + c_1 w_1 + c_k w_k = 0$

$c_1 w_1 + \dots c_k w_k = -a_1 u_1 - \dots - a_m u_m - b_1 v_1 - \dots b_j v_j \in V_1$

since $w_1, \dots, w_k \in V_2$,

$c_1 w_1 + \dots + c_k w_k \in V_2$

so $c_1 w_1 + \dots c_k w_k \in V_1 \cap V_2$

$c_1 w_1 + \dots + c_k w_k = d_1 u_1 + \dots + d_m u_m$ for some $d_1, \dots, d_m \in \mathbb{F}$

$c_1 w_1 + \dots c_k w_k - d_1 u_1 - \dots - d_m u_m = 0$

because $u_1, \dots u_m, w_1, \dots w_m$ is ℓI,

$c_1 = 0$, $c_k = 0$, $d_1 = 0$, $d_m = 0$

Original equation reduces to:

$a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_j v_j = 0$

$a_1 = 0, \dots a_m = 0, b_1 = 0 \dots b_j = 0$ All scalars are zero

so $u_1, \dots u_m, v_1, \dots v_j, w_1, \dots, w_k$ is ℓI.

By 2.39, it's a basis of $V_1 + V_2$

$\dim(V_1 + V_2) = m + j + k = (m+j) + (m+k) - m = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$ ∎