2.28 Example

The list $(1,-1,0), (1,0,-1)$ is a basis of $\{(x,y,z)\in\mathbb{F}^3 :$
$x+y+z=0\}$

$(x,y,z) = (x,y,-x-y)$

$= x(1,0,-1) + y(0,1,-1)$

$\therefore (1,0,-1), (0,1,-1)$ spans U.

Suppose $a,b\in\mathbb{F}$ satisfy

$a(1,0,-1) + b(0,1,-1) = (0,0,0)$

$(a, b, -a-b) = 0,0,0$

$\therefore\ a=b=0$

$\therefore$ the list $(1,0,-1), (0,1,-1)$ is L.I.

$\therefore (1,0,-1), (0,1,-1)$ is a basis of U.

- The list $1, z, \dots, z^m$ is a basis of $P_m(\mathbb{F})$.

  Every polynomial $p\in P_m(\mathbb{F})$ is written

  $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$,

  for all $z\in\mathbb{C}$, for all $a_0, a_1, a_2, \dots, a_m \in\mathbb{F}$.

  Notice that $p(z)$ is a linear combination of $1, z, z^2, \dots, z^m$.

  Also, for example,

  $1, z+3, (z+3)^2, \dots, (z+3)^m$ is a basis of $P_m(\mathbb{F})$

  Another example,

  $1, z-6, (z-6)^2$ is a basis of $P_2(\mathbb{F})$

2C Dimension

Recall 2.23 from 2B

2.23: Length of linearly independent list $\leq$ length of spanning list in a finite-dimensional vector space.

2.35: The length of a basis of a vector space does not depend on the basis

Any two bases of a finite-dimensional vector space have the same length (same number of vectors in the bases)

Proof:

Suppose V is a finite-dimensional vector space.

Let $B_1 = v_1, \ldots, v_m$ and $B_2 = w_1, \ldots, w_n$ be two bases of V.

Then, by 2.23 of Axler, the length of B_1 is less than or equal to the length of B_2.

Interchange (swap) the roles of B_1 and B_2. By 2.23 Axler, the length of B_2 is less than or equal to the length of B_1.

In other words, we have

length of $B_1 \leq$ length of B_2

and, length of $B_2 \leq$ length of B_1

$\therefore$ length of $B_2 =$ length of B_1

$\therefore$ the two bases have the same length.

2.36 Definition

The dimension of a finite-dimensional vector space V is the length of any basis B of V. (Denoted $\dim V$)

2.37 Example

- $\dim \mathbb{F}^n = n$ because the length of any basis of $\mathbb{F}^n$

is n (any basis of $\mathbb{F}^n$ contains n elements).

- $\dim P_m(\mathbb{F}) = m+1$ because, for example, $1, z, z^2, \ldots, z^m$ is a basis of $P_m(\mathbb{F})$ and the length of the basis is $m+1$.

2.38 Dimension of a subspace.

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$

Proof:

Since V is finite-dimensional and U is a subspace of V, by 2.26 Axler, U is also finite-dimensional. By 2.32 of Axler, there exist a basis of U and a basis of V. This means in particular, that the basis of U is a linearly independent list in V and the basis of V is a spanning list of V.

Recall from 2.23: Length of linearly independent list $\leq$ length of spanning list.

The length of our linearly independent list in U is $\dim V$. Likewise, the length of our spanning list in V is $\dim V$, $\therefore \dim U \leq \dim V$.

2.39 Linearly independent list of the right length is a basis

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Proof: Suppose $\dim V = n$. Let $v_1, \ldots, v_n$ be a linearly

independent list in V. By 2.33 of Axler, we can extend $v_1, \dots, v_n$, if necessary to a basis of V.
But every basis of V has length n. Since $v_1, \dots, v_n$ already has length n, in this case, we do not need to extend to a basis of V. This means, $v_1, \dots, v_n$ is itself is a basis of V.

2.40 Example

Show that the list (5,7), (4,3) is a basis of $\mathbb{F}$?

Proof:

We will show that (5,7), (4,3) is linearly independent.

Suppose $a_1, a_2 \in \mathbb{F}$ satisfy

$$a_1(5,7) + a_2(4,3) = (0,0)$$

$$\Rightarrow (5a_1 + 4a_2,\ 7a_1 + 3a_2) = (0,0)$$

$$\Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \end{cases}$$

$\therefore$ (5,7), (4,3) is linearly independent in $\mathbb{F}^2$

Since (5,7), (4,3) has length 2 and $\dim \mathbb{F}^2 = 2$, by 2.39 of Axler, we conclude that (5,7), (4,3) is a basis of $\mathbb{F}^2$.

2.41 Example

Show that $1, (x-5)^2, (x-5)^3$ is a basis of the subspace U of $P_3(\mathbb{R})$ defined by

$$U = \{p \in P_3(\mathbb{R}) : p'(5) = 0\}$$

Proof: Let $p_1(x)=1$. Then $p_1'(x)=0$ so $p_1'(5)=0$
and so $p_1=1\in U$.
Let $p_2(x)=(x-5)^2$. Then $p_2'(x)=2(x-5)$
$\therefore p_2'(5)=0$, and so $p_2=(x-5)^2\in U$.
Let $p_3(x)=(x-5)^3$. Then $p_3'(x)=3(x-5)^2$
$\therefore p_3'(5)=0$, and so $p_3=(x-5)^3\in U$
$\therefore 1, (x-5)^2, (x-5)^3\in U$.

- Next, suppose $a,b,c\in\mathbb{R}$ satisfy
$$a+b(x-5)^2+c(x-5)^3=0 \quad \text{for all } x\in\mathbb{R}$$
Notice that the left-hand side of the above equation contains the cx^3 term, but the right hand side not
$\therefore c=0$
LHS has bx^2 term, RHS not, $\therefore b=0$
Since $b=0$, $c=0$ $\therefore$ implies $a=0$
So $1, (x-5)^2, (x-5)^3$ is linearly independent.
Since the length of $1, (x-5)^2, (x-5)^3$ is 3.
and $\dim U=3$ by 2.39.

- Note that $\dim U$ is at most 4, but it cannot equal 4 because if $\dim U=4$, then we can extend a basis of U to a basis of $P_3(\mathbb{R})$, which would produce a list with length greater than 4
So $\dim U=3$.

2.42 Spanning list of the right length is a basis.
Suppose V is a finite-dimensional vector space. Then

every spanning list of vectors in V with length $\dim V$ is a basis of V.

Proof:

Suppose $\dim V = n$ and $v_1, \ldots, v_n$ spans V. By 2.31 Axler, we can reduce to a basis of V. But every basis of V has length n, so in this case we do not need to reduce anything; we do not need to remove any elements of $v_1, \ldots, v_n$. Therefore, $v_1, \ldots, v_n$ itself is a basis of V.

2.43 Dimension of a sum

If U_1 and U_2 are subspaces of a finite-dimensional vector space V, then

$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$

Proof:

Let $m = \dim(U_1 \cap U_2)$ and let $u_1, \ldots, u_m$ be a basis of $U_1 \cap U_2$. Then it is linearly independent in U_1. So by 2.33, we can extend this list to a basis $u_1, \ldots, u_m, v_1, \ldots, v_j$ of U_1, which means $\dim U_1 = m + j$. Similarly, $u_1, \ldots, u_m$ is linearly independent in U_2, so by 2.33, we can extend this list to a basis $u_1, \ldots, u_m, w_1, \ldots, w_k$ of U_2, which means $\dim U_2 = m + k$.
We will prove that the list $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is a basis of $U_1 + U_2$. We have $U_1, U_2 \subset \operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$ which means

$\operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k) = U_1 + U_2$

So the dimensions of $U_1 + U_2$ and $\operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$ are equal. If $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is L.I. by 2.39, it would be a basis.

Discussion Notes.

Re-explain Example 2.41

Why does $U = \{p \in P_3(\mathbb{R}) : p'(5) = 0\}$ have dimension 3 ?

Since we proved $1, (x-5)^2, (x-5)^3$ is L.I. in U.

$\dim U$ is 3 or 4.

Since U is a subspace of $P_3(\mathbb{R})$, by 2.38 of Axler.

$3 \leq \dim V \leq \dim P_3(\mathbb{R}) = 4$ (if U is a subspace of V, then $\dim U \leq \dim V$).

However, $q = x-5 \in P_3(\mathbb{R})$ But $q = x-5 \notin U$ because

$q'(5) = 1$ ($q'(5) \neq 0 \therefore q \notin U$)

We found a polynomial such as $x-5$ that is in $P_3(\mathbb{R})$ but not in U.

Therefore, $U \neq P_3(\mathbb{R})$

This means we have

$$3 \leq \dim U < \dim P_3(\mathbb{R}) = 4$$

So we conclude $\dim U = 3$

2.43 Prove

$u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is L.I.

Suppose $a_1u_1 + \ldots + a_mu_m + b_1v_1 + \ldots + b_jv_j + c_1w_1 + \ldots + c_kw_k = 0$

Need to prove $a_1 = \ldots = a_m = b_1 = \ldots = b_j = c_1 = \ldots = c_k = 0$

Since $u_1, \ldots, u_m, v_1, \ldots v_j$ is a basis of U. we have

$c_1w_1 + \ldots + c_kw_k = -a_1u_1 - \ldots - a_mu_m - b_1v_1 - \ldots - b_jv_j \in U_1$

Since $c_1w_1 + \ldots + c_kw_k \in U$, we have.

$c_1w_1 + \cdots + c_kw_k \in U_2$

$\therefore c_1w_1 + \dots + c_kw_k \in U_1 \cap U_2$

Since we introduced $u_1, \dots, u_m$ to be a basis of $U_1 \cap U_2$

we can write $c_1w_1 + \dots + c_kw_k = d_1u_1 + \dots + d_mu_m$

for some $d_1, \dots, d_m \in \mathbb{F}$. This means,

$c_1w_1 + \dots + c_kw_k - d_1u_1 - \dots - d_mu_m = 0$

Since $u_1, \dots, u_m, w_1, \dots, w_k$ is linearly independent, all the scalars are zero;

$$c_1 = 0, \dots, c_k = 0, d_1 = 0, \dots, d_m = 0$$

In particular, $c_1 = 0, \dots, c_k = 0$

So, the original eq.

$$a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_jv_j + c_1w_1 + \dots + c_kw_k = 0$$

reduces to

$$a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_jv_j = 0$$

Since $u_1, \dots, u_m, v_1, \dots, v_j$ is a basis of U_1, it is linearly independent, so

$a_1 = 0, \dots, a_m = 0, b_1 = 0, \dots, b_j = 0$

So all scalars are zero.

So $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is LI.

By 2.39, it is also a basis of $U_1 + U_2$

$\therefore$ We have
$$\begin{aligned} \dim(U_1 + U_2) &= m + j + k \\ &= (m+j) + (m+k) - m \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) \end{aligned}$$