7/3 2.C Dimension

We need to recall 2.23 of Axler from Section 2.B

2.23: length of linearly independent list
$\leq$ length of spanning list
in a finite-dimensional vector space

2.35 The length of a basis of a vector space does not depend on the basis

$v_1, \ldots, v_m, w_1, \ldots, w_n \in V$

Any two bases of a finite-dimensional vector space have the same length (same number of vectors in the basis).

Proof: Suppose V is a finite-dimensional vector space.
Let $B_1 = v_1, \ldots, v_m$ and $B_2 = w_1, \ldots, w_n$ be two bases of V.
Then by 2.23 of Axler, the length of B_1 is less than or equal to length of B_2.

Interchange (swap) the roles of B_1 and B_2. By 2.23 of Axler, length of B_2 is less than or equal to the length of B_1.

In other words, we have

length of $B_1 \leq$ length of B_2

and

length of $B_2 \leq$ length of B_1

Therefore

length of $B_1 =$ length of B_2.

In other words, the two bases have the same length. $\boxtimes$

2.36 Definition:

The dimension of a finite dimensional vector space V is the length of any basis B of V.

The dimension of V is denoted dim V.

2.37 Example!

- dim $\mathbb{F}^n = n$ because the length of any basis of $\mathbb{F}^n$ is n (any bases of $\mathbb{F}^n$ contains n elements).
- dim $\mathcal{P}_m(\mathbb{F}) = m+1$ b/c, for example, $1, z, z^2, .., z^m$ is a basis of $\mathcal{P}_m(\mathbb{F})$ and the length of the basis is $m+1$.

2.38 Dimension of a Subspace

If V is finite dimensional and U is a subspace of V, then dim $U \leq$ dim V.

Proof: Since V is finite dimensional and U is a subspace of ~~V~~ V, by 2.26 of Axler, U is also finite-dimensional. By 2.32 of Axler, there exist a basis of U and a basis of V. This means in particular, that the basis of U is a linearly independent list in V and the basis of V is a spanning list of V.

Recall from 2.23 of Axler

length of linearly independent list $\leq$ length of spanning list.

The length of our linearly independent list in V is $\dim V$ because, the length of our spanning list in V is $\dim V$.

Therefore, $\dim V \leq \dim V$. $\boxed{\checkmark}$

Useful result!

2.39 Linearly independent list of the right length is a basis

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Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Proof: Suppose $\dim V = n$. Let $v_1, \ldots, v_n$ be a linearly independent list in V. By 2.33 of Axler, we can extend $v_1, \ldots, v_n$, (if necessary) to a basis of V. But every basis of V has length n. Since $v_1, \ldots, v_n$ already has length n, in this case, we do NOT need to extend to a basis of V. This means, $v_1, \ldots, v_n$ is itself a basis of V. $\boxed{\checkmark}$

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2.40 Example

Show that the list $(5,7),(4,3)$ is a basis of $\mathbb{F}^2$.

Proof: We will show that $(5,7),(4,3)$ is linearly independent.

Suppose $a_1, a_2 \in \mathbb{F}$ satisfy

$$a_1(5,7)+a_2(4,3)=(0,0)$$

Then we have

$$(5a_1+4a_2, 7a_1+3a_2)=(0,0)$$

Equate the coordinates to get the system of equations

$$5a_1+4a_2=0$$
$$7a_1+3a_2=0$$

System-solve to get

$$a_1=0, a_2=0$$

So the list $(5,7),(4,3)$ is linearly independent in $\mathbb{F}^2$.
Since $(5,7),(4,3)$ has length 2 and $\dim \mathbb{F}^2=2$,
by 2.39 of Axler, we conclude that $(5,7),(4,3)$
is a basis of $\mathbb{F}^2$.

2.41 Example 1

Show that $1,(x-5)^2,(x-5)^3$ is a basis of the subspace
U of $\mathcal{P}_3(\mathbb{R})$ defined by

$$U=\{p \in \mathcal{P}_3(\mathbb{R}): p'(5)=0\}$$

Proof:

Let $p_1(x)=1$. then $p_1'(x)=0$, so $p_1'(5)=0$ and
so $p_1=1\in U$.

Let $p_2(x)=(x-5)^2$. Then $p_2'(x)=2(x-5)$.
So $p_2'(5)=0$, and so $p_2=(x-5)^2\in U$.

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Let $p_3(x) = (x-5)^3$. Then $p_3'(x) = 3(x-5)^2$.
So $p_3'(5) = 0$, and so $p_3 = (x-5)^3 \in U$.
Therefore, $1, (x-5)^2, (x-5)^3 \in U$.

Next, suppose $a, b, c \in \mathbb{R}$ satisfy
$$a + b(x-5)^2 + c(x-5)^3 = 0 \quad \text{for all } x \in \mathbb{R}.$$
Notice that the LHS of the above equation contains the cx^3 term, but the RHS does not. So $c = 0$.

Similarly, the LHS contains the bx^2 term, but the RHS does not. So $b = 0$
Since $b = 0$ and $c = 0$, the above equation implies $a = 0$.

So $1, (x-5)^2, (x-5)^3$ is linearly independent
since the length of $1, (x-5)^2, (x-5)^3$ is 3 and $\dim V = 3$, by 2.39
$1, (x-5)^2, (x-5)^3$ is a basis of V.

Note that $\dim V$ is at most 4, but it cannot equal 4. b/c if $\dim V = 4$, then we can extend a basis of V to a basis of $\mathcal{P}_3(\mathbb{R})$, which would produce a list w/ length greater than 4. So $\dim V = 3$. (p.40)

2.42 Spanning list of the right length is a basis

Suppose V is a finite-dimensional vector space. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

<u>Proof</u>: Suppose $\dim V = n$ and $v_1, \ldots, v_n$ spans V. By 2.31 of Axler, we can reduce (if necessary) to a basis of V. But every basis of V has length n, so in this case we do not need to reduce anything; we do not need to remove any elements of $v_1, \ldots, v_n$. Therefore, $v_1, \ldots, v_n$ itself is a basis of V. $\boxtimes$

2.43 Dimension of a sum

If U_1 and U_2 are subspaces of a finite-dimensional vector space V, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

<u>Proof</u>: Let $m = \dim(U_1 \cap U_2)$ and let $u_1, \ldots, u_m$ be a basis of $U_1 \cap U_2$. Then it is linearly independent in U_1. So by 2.33 of Axler, we can extend this list to a basis $u_1, \ldots, u_m, v_1, \ldots, v_j$ of U_1, which means $\dim U_1 = m + j$. Similarly, $u_1, \ldots, u_m$ is linearly independent in U_2, so, by 2.33 of Axler, we can extend this list to a basis $u_1, \ldots, u_m, w_1, \ldots, w_k$ of U_2, which means $\dim U_2 = m + k$.

We will prove that the list $u_1, .., u_m, v_1, .., v_j, w_1, .., w_k$ is a basis of $V_1 + V_2$. We have $V_1, V_2 \subset \text{span}(u_1, .., u_m, v_1, .., v_j, w_1, .., w_k)$ which means $\text{Span}(u_1, .., u_m, v_1, .., v_j, w_1, .., w_k) = V_1 + V_2$. So the dimensions of $V_1 + V_2$ and $\text{span}(u_1, .., u_m, v_1, ..., v_j, w_1, .. w_k)$ are equal. If $u_1, .., u_m, v_1, .. v_j, w_1, .., w_k$ is linearly independent, then by 2.35 of Axler it would be a basis.

Prove that $u_1, .., u_m, v_1, .., v_j, w_1, .. w_k$ is lin. independent

7/3 DISCUSSION

Re-explain Example 2.41. Why does $U = \{p = P_3(\mathbb{R}) : p'(5) = 0\}$ have dimension 3?

Since we proved $1, (x-5)^2, (x-5)^3$ is linearly independent in U, dim U is 3 or 4.

Since U is a subspace of $P_3(\mathbb{R})$, by 2.38 of Axler $3 \leq \dim U \leq \dim P_3(\mathbb{R}) = 4$ (if U is a subspace of V, then $\dim U \leq \dim V$)

However, $q = x-5 \in P_3(\mathbb{R})$, But $q = x-5 \notin U$ because

$q'(5) = 1$ $\qquad$ $q'(x) = 1$

$(q'(5) \neq 0)$

So $q \notin U$

We found a polynomial such as $x-5$ that is in $P_3(\mathbb{R})$ but not in U. Therefore $U \neq P_3(\mathbb{R})$

This means we have

$$3 \leq \dim U < \dim P_3(\mathbb{R}) = 4$$

So we conclude $\dim U = 3$.

2.43 Prove $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is linearly independent

Suppose

$$a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_jv_j + c_1w_1 + \dots + c_kw_k = 0$$

Need to prove:

$$a_1 = 0, \dots, a_m = 0, b_1 = 0, \dots, b_j = 0, c_1 = 0, \dots, c_k = 0.$$

Since $u_1, \dots, u_m, v_1, \dots, v_j$ is a basis of U_1, we have

$$c_1w_1 + \dots + c_kw_k = -a_1u_1 - \dots - a_mu_m - b_1v_1 - \dots - b_jv_j \in U_1.$$

Since $w_1, \dots, w_k \in U_2$, we have

$$c_1w_1 + \dots + c_kw_k \in U_2.$$

So

$$c_1w_1 + \dots + c_kw_k \in U_1 \cap U_2.$$

Since we introduced $u_1, \dots, u_m$ to be a basis of $U_1 \cap U_2$, we can write $c_1w_1 + \dots + c_kw_k = d_1u_1 + \dots + d_mu_m$ for some $d_1, \dots, d_m \in \mathbb{F}$. This means

$$c_1w_1 + \dots + c_kw_k - d_1u_1 - \dots - d_mu_m = 0.$$

Since $u_1, \dots, u_m, w_1, \dots, w_k$ is linearly independent, all scalars are zero; $c_1 = 0, \dots, c_k = 0, d_1 = 0, \dots, d_m = 0$

In particular

$$c_1 = 0, \dots, c_k = 0$$

See the original eq.
$a_1 u_1 + .. + a_m u_m + b_1 v_1 + .. + b_j v_j + c_1 w_1 + .. + c_k w_k = 0$
reduces to
$a_1 u_1 + .. + a_m u_m + b_1 v_1 + .. + b_j v_j = 0$

Since $u_1, .., u_m, v_1, .., v_j$ is a basis of V_1, it is linearly independent, so
$a_1 = 0, .., a_m = 0, b_1 = 0, ..., b_j = 0$
So all scalars are zero.

So $u_1, .., u_m, v_1, .., v_j, w_1, .., w_k$ is linearly independent.
Then by 2.39 of Axler, it is also a basis of $V_1 + V_2$.
Therefore, we have

$$\begin{aligned} \dim(V_1+V_2) &= m+j+k \\ &= (m+j)+(m+k)-m \\ &= \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2) \end{aligned}$$

$\boxtimes$

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