2C Dimension

We need to recall 2.23 of Axler from sec. 2B

Defn 2.23 Length of Linearly Independent List $\leq$ length of spanning list in a finite-dimensional vector space.

Defn 2.35 Length of a basis of a vector space does not depend on the basis

Any two bases of a finite-dimensional vector space have the same length (same number of vectors in the bases)

proof: Suppose V is a finite-dimensional vector space. Let $B_1 = v_1, \dots, v_m$ and $B_2 = w_1, \dots, w_n$ be two bases of V. then by 2.33 of Axler, the length of B_1 is less than or equal to the length of B_2.

Interchange (swap) the roles of B_1 and B_2. By 2.23 of Axler, the length of B_2 is less than or equal to the length of B_1.

In other words, we have

length of $B_1 \leq$ length of B_2

and length of $B_2 \leq$ length of B_1

Therefore, length of B_1 = length of B_2

In other words, the two bases have the same length. ◻

Defn 2.36 The dimension of a finite-dimensional vector space V is the length of any basis B of V

The dimension of V is denoted $\dim V$.

2.37 Eg • $\dim \mathbb{F}^n = n$ because the length of any bases of $\mathbb{F}^n$ is n
(any bases of $\mathbb{F}^n$ contains n elements)
• $\dim \mathcal{P}_m(\mathbb{F}) = m+1$ because, for example
Eg $1, z, z^2, \ldots, z^m$ is a basis of $\mathcal{P}_m(\mathbb{F})$ and the length of the basis is, $m+1$.

Defn 2.38 Dimension of a Subspace
If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

proof: Since V is finite-dimensional and U is a subspace of V, by 2.26 of Axler, U is also finite-dimensional. By 2.32 of Axler, There exists a basis of U and a basis of V. This means in particular that the basis of U is a linearly indepent list in U and the basis of V is a spanning list of V.
↳ Recall from 2.23 of Axler: Length of linearly independent list $\leq$ length of spanning lis.
The length of our linearly independent list in U, is $\dim U$. Likewise, the length of our spanning list in V is $\dim V$.
Therefor, $\dim U \leq \dim V$. ☑

Defn 2.39 Linearly independent list of the right length is a basis
useful result!
Suppose V is a finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

proof →

proof: Suppose $\dim V = n$. Let $v_1, \cdots, v_n$ be a linearly independent list in V. By 2.33 of Axler, we can extend $v_1, \ldots, v_n$ to a basis of V. (if necessary)

But every basis of V has length n. Since $v_1, \cdots, v_n$ already has length n, in this case, we do NOT need to extend to a basis of V. This means, $v_1, \cdots, v_n$ is itself a basis of V. $\boxtimes$

2.40 Eg Show that the list $(5,7), (4,3)$ is a basis of $\mathbb{F}^2$?

proof. We will show that $(5,7), (4,3)$ is linearly independent. Suppose $a_1, a_2 \in \mathbb{F}$ satisfy

$$a_1(5,7) + a_2(4,3) = (0,0)$$

Then we have $(5a_1 + 4a_2, 7a_1 + 3a_2) = (0,0)$

Equate the coordinates ~~to~~ to get the system of equations

$$5a_1 + 4a_2 = 0$$
$$7a_1 + 3a_2 = 0$$

... System-solve to get

$$a_1 = 0, \quad a_2 = 0$$

So the list $(5,7), (4,3)$ is linearly independent in $\mathbb{F}^2$.

Since $(5,7), (4,3)$ has length 2 and $\dim \mathbb{F}^2 = 2$, by 2.39 of Axler, we conclude that $(5,7), (4,3)$ is a basis of $\mathbb{F}^2$. $\boxtimes$

2.41 Eg Show that $1, (x-5)^2, (x-5)^3$ is a basis of the subspace U of $\mathcal{P}_3(\mathbb{R})$ defined by

$$U = \{p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0\}$$

proof:
- Let $p_1(x) = 1$, Then $p_1'(x) = 0$, so $p_1'(5) = 0$ and so $p_1 = 1 \in U$
- Let $p_2(x) = (x-5)^2$. Then $p_2'(x) = 2(x-5)$. →

so $p_2'(5)=0$, and so $p_2'=(x-5)^2 \in O$

- Let $p_3(x)=(x-5)^3$. Then $p_3'(x)=3(x-5)^2$.
  so $p_3'(5)=0$ and so $p_3=(x-5)^3 \in U$.
- Therefore $1, (x-5)^2, (x-3)^3 \in U$
- Next, suppose $a, b, c \in \mathbb{R}$ satisfy
  $$a+b(x-5)^2+c(x-5)^3=0 \quad \text{for all } x \in \mathbb{R}$$
  - Notice that the left-hand side of the above equation contains the cx^3 term, but the right-hand side does not. So $c=0$.
  - Similarly, the left-hand side contains the bx^2 term, but the right-hand side does not. so $b=0$
  - Since $b=0$ and $c=0$, the above equation implies $a=0$.

Defn 2.42 Spanning List of the Right length is a basis
Suppose V is a finite-dimensional vector space. Then every spanning list of vectors in V with length $\dim V$ is a basis of V. see proof

So $1, (x-5)^2, (x-5)^3$ is linearly independent since the length of $1, (x-5)^2, (x-5)^3$ is 3 and $\boxed{\dim U = 3}$, by 2.39 $1, (x-5)^2, (x-5)^3$ is a basis of V.

- Note (see pg 46 of Axler) that $\dim U$ is at most 4, but it cannot equal 4 because if $\dim U=4$, then we can extend a basis of U to a basis of $\mathcal{P}_3(\mathbb{R})$, which would produce a list with length greater than 4. So $\dim U = 3$

See Defn 2.42

proof: Suppose $\dim V = n$ and $v_1, \dots, v_n$ spans V. By 2.31 of Axler, we can reduce (if necessary) to a basis of V. But every basis of V has length n, so in this case we do not need to remove any elements of $v_1, \dots, v_n$.
Therefore, $v_1, \dots, v_n$ is itself a basis of V. $\boxed{=}$

Thm 2.43 Dimension of Sum
If U_1 and U_2 are subspaces of a finite-dimensional vector space V, then
$$\dim(U_1+U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

proof: Let $m = \dim(U_1 \cap U_2)$, and let $u_1, \dots, u_m$ be a basis of $U_1 \cap U_2$. Then it is linearly independent in U_1. So, by 2.33 of Axler, we can extend this list to a basis $u_1, \dots, u_m, v_1, \dots, v_j$ of U_1, which means $\dim U_1 = m + j$. Similarly, $u_1, \dots, u_n$ is linearly independent in U_2. So, by 2.33 of Axler, we can extend this list to a basis $u_1, \dots, u_m, w_1, \dots, w_k$ of U_2, which means $\dim U_2 = m+k$.

- We will prove that the list $u_1, \dots, u_m, (v_1, \dots, v_j,) w_1, \dots, w_k$ is a basis of $U_1 + U_2$. We have
$$U_1, U_2 \subset \operatorname{span}(u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k) = U_1 + U_2$$
So the dimensions of $U_1 + U_2$ and $\operatorname{Span}(u_1, \dots u_m, v_1, \dots v_j, w_1, \dots w_k)$ are equal. If $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is linearly independant, then by 2.39 of Axler it would be a basis.
- Prove that $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is lin independent.
Suppose $a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_jv_j + c_1w_1 + \dots + c_kw_k = 0$.
Need to prove: $a_1 = 0, \dots, a_m = 0, b_1 = 0, \dots, b_j = 0, c_1 = 0, \dots, c_k = 0$
Since $u_1, \dots, u_m, v_1, \dots, v_j$ is a basis of U_1, we have
$\Rightarrow c_1w_1 + \dots + c_kw_k = -a_1u_1 - \dots - a_mu_m - b_1v_1 - \dots - b_jv_j \in U_1$

Disc. 7/3
Re-explain
2.41

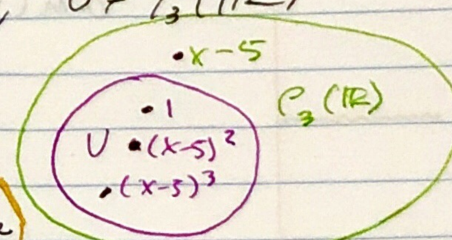
Eg why does
$U = \{ p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0 \}$ have dimension 3?

Since we prove $1, (x-5)^2, (x-5)^3$ is linearly independent in U, $\dim U$ is 3 or 4.

Since U is a subspace of $\mathcal{P}_3(\mathbb{R})$, by 2.38 of Axler
$3 \leq \dim U \leq \dim \mathcal{P}_3(\mathbb{R}) = 4$ (if U is a subspace of V, then $\dim U \leq \dim V$).

However, $q = x-5 \in \mathcal{P}_3(\mathbb{R})$, BUT $q = x-5 \notin U$ because $q'(x) = 1$
$q'(5) = 1$
$(q'(5) \neq 0)$
So $q \notin U$.

We found a polynomial such as $x-5$ that is in $\mathcal{P}_3(\mathbb{R})$ but not in U. Therefore, $U \neq \mathcal{P}_3(\mathbb{R})$



This means we have
$3 \leq \dim U < \dim \mathcal{P}_3(\mathbb{R}) = 4$
So we conclude $\dim U = 3$

Lecture notes cont:

Since $w_1, \dots, w_k \in U_1$, we have
$c_1 w_1 + \dots + c_k w_k \in U_2$.
So, $c_1 w_1 + \dots + c_k w_k \in U_1 \cap U_2$
Since we introduced $u_1, \dots, u_m$ to be a basis of $U_1 \cap U_2$, we can write
$$c_1 w_1 + \dots + c_k w_k = d_1 u_1 + \dots + d_m u_m$$
for some $d_1, \dots, d_m \in \mathbb{F}$. This means
$$c_1 w_1 + \dots + c_k w_k - d_1 u_1 - \dots - d_m u_m = 0.$$
Since $u_1, \dots, u_m, w_1, \dots w_k$ is linearly independent, all the scalars are zero
$$c_1 = 0, \dots, c_k = 0, d_1 = 0, \dots, d_m = 0.$$
In particular
$$c_1 = 0, \dots, c_k = 0$$

cont.

So the original eq.

$$a_1u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots b_j v_j + c_1 w_1 + \cdots c_k w_k = 0$$

reduces to

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j = 0$$

Since $u_1, \ldots, u_m, v_1, \ldots, v_j$ is a basis of U_1, it is linearly independent, so

$$a_1 = 0, \ldots a_m = 0, b_1 = 0, \ldots, b_j = 0.$$

So all scalars are zero.

So $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots w_k$ is linearly indep.
Then by 2.39 of Axler, it is also a basis of $U_1 + U_2$.

Therefore we have

$$\begin{aligned} \dim(U_1 + U_2) &= m + j + k \\ &= (m+j) + (m+k) - m \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) \quad \boxed{} \end{aligned}$$

Disc. 7/3
Re-explain
2.41

Eg why does
$U = \{p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0\}$ have dimension 3?

Since we prove $1, (x-5)^2, (x-5)^3$ is linearly independent in U, $\dim U$ is 3 or 4.

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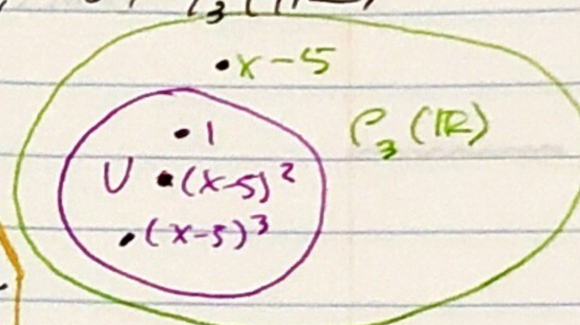
$q'(5) = 1$ $\qquad q'(x) = 1$
($q'(5) \neq 0$)
So $q \notin U$.

We found a polynomial such as $x-5$ that is in $\mathcal{P}_3(\mathbb{R})$ but not in U. Therefore, $U \neq \mathcal{P}_3(\mathbb{R})$
This means we have
$3 \le \dim U < \dim \mathcal{P}_3(\mathbb{R}) = 4$
So we conclude $\dim U = 3$



Lecture notes cont:

Since $w_1, \dots, w_k \in U_1$, we have
$c_1 w_1 + \dots + c_k w_k \in U_2$.

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