Proof: Next, we will show $U \cap W = \{0\}$

Suppose $v \in U \cap W$. Then there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in F$ such that $v = a_1u_1 + \ldots + a_mu_m = b_1w_1 + \ldots + b_nw_n$

subtracting we get

$$a_1v_1 + \ldots + a_mv_m - b_1w_1 - \ldots - b_nw_n = 0$$

Since $v_1, \ldots, v_m, w_1, \ldots, w_n$ are linearly indep., all the scalars are zero

$$a_1 = 0, \ldots, a_m = 0, -b_1 = 0, \ldots, -b_n = 0.$$
$$(b_1 = 0, \ldots, b_n = 0).$$

Therefore, $v = a_1v_1 + \ldots + a_mv_m$
$= 0u_1 + \ldots + 0u_m$
$= 0 \in \{0\}$

and $v = b_1w_1 + \ldots + b_nw_n$
$= 0w_1 + \ldots + 0w_n$
$= 0 \in \{0\}$

Therefore, $U \cap W \subset \{0\}$

so $U \cap W = \{0\}$, as desired

2.C Dimension

We need to recall 2.23 of Axler section 2.B

2.23: Length of lin indep. list $\leq$ length of spanning list in a finite-dimensional vector space

2.35 The length of a basis of a vector space does not depend on the basis

Any two bases of a finite dimensional vector space have the same length (same number of vectors in the bases)

Proof: Suppose V is a finite dim vector space. Let $B_1 = v_1, \ldots, v_n$ and $B_2 = w_1, \ldots, w_n$ be two bases of V. Then, by 2.23 of Axler, the length of B_1 is less than or equal to the length of B_2.

$B_1 = v_1, \ldots, v_m$
$B_2 = w_1, \ldots, w_n$
$v_1, \ldots, v_m, w_1, \ldots, w_n$

Interchange (swap) the roles of B_1 & B_2. By 2.23 of Axler, the length of B_2 is $\leq$ the length of B_1.

In other words, we have length of $B_1 \leq$ length of B_2 and length of $B_2 \leq$ length of B_1

Therefore, length of B_1 = length of B_2.

In other words, the two bases have the same length.

2.36 Definition:

The Dimension of a finite-dimensional vector space V is the length of any basis B of V.

The dimension of V is denoted $\dim V$.

2.37 Example

- $\dim \mathbb{F}^n = n$ b/c the length of any basis of $\mathbb{F}^n$ is n (any basis of $\mathbb{F}^n$ contains n elements)
- $\dim P_m(\mathbb{F}) = m+1$ because, for example, $1, z, z^2, \dots, z^m$ is a basis of $P_m(\mathbb{F})$ and the length of the basis is $m+1$.

2.38 Dimension of a subspace

If V is finite dimensional & U is a subspace of V, then $\dim U \leq \dim V$

Proof: Since V is finite dim. & U is a subspace of V, by 2.26 of Axler U is also finite-dim. By 2.32 of Axler, there exist a basis of U and a basis of V. This means in particular, that the basis of U is a lin indep. list in U & the basis of V is a spanning list of V. Recall from 2.23 of Axler: length of lin indep. list $\leq$ length of spanning list.

The length of our lin indep. list in U is $\dim U$. Likewise, the length of our spanning list in V is $\dim V$. Therefore, $\dim U \leq \dim V$

Useful Result!

2.39 Lin. indep. list of the right length is a basis

Suppose V is finite-dim. Then every lin. indep. list of vectors in V w/ length $\dim V$ is a basis of V.

Proof: Suppose $\dim V = n$. Let $v_1, \dots, v_n$ be a lin indep. list in V. By 2.33 of Axler, we can extend $v_1, \dots, v_n$ to a basis of V (if necessary). But every basis of V has length n. Since $v_1, \dots, v_n$ already has length n in this case, we do not need to extend to a basis of V. This means, $v_1, \dots, v_n$ is itself a basis of V.

2.40 Example

Show that the list $(5,7), (4,3)$ is a basis of $\mathbb{F}^2$.

Proof: We will show that $(5,7), (4,3)$ is lin indep.

Suppose $a_1, a_2 \in \mathbb{F}$ satisfy $a_1(5,7) + a_2(4,3) = (0,0)$

Then we have, $(5a_1 + 4a_2, 7a_1 + 3a_2) = (0,0)$

Equate the coordinates to get the system of eqths.

$$5a_1 + 4a_2 = 0$$
$$7a_1 + 3a_2 = 0$$

System solve to get $a_1 = 0$, $a_2 = 0$

So the list $(5,7), (4,3)$ is lin indep. in $\mathbb{F}^2$

Since $(5,7), (4,3)$ has length of 2 & $\dim \mathbb{F}^2 = 2$, by 2.39 of Axler, we conclude that (5,7), (4,3) is a basis of $\mathbb{F}^2$

2.41 Example

Show that $1, (x-5)^2, (x-5)^3$ is a basis of the subspace U of $P_3(\mathbb{R})$ defined by
$U = \{p \in P_3(\mathbb{R}) : p'(5) = 0\}$

Proof: Let $p_1(x) = 1$. Then $p_1'(x) = 0$, so $p_1'(5) = 0$.
So $p_1 = 1 \in U$
Let $p_2(x) = (x-5)^2$. Then $p_2' = (x-5)$
so $p_2'(5) = 0$ and so $p_2 = (x-5)^2 \in U$.
Let $p_3(x) = (x-5)^3$. Then $p_3'(x) = (x-5)^2$
so $p_3'(5) = 0$, and so $p_3 = (x-5)^3 \in U$
Therefore, $1, (x-5)^2, (x-5)^3 \in U$.

Next, suppose $a, b, c \in \mathbb{R}$ satisfy $a + b(x-5)^2 + c(x-5)^3 = 0$ for all $x \in \mathbb{R}$. Notice that left-hand side of the above eqth contains the cx^3 term, but the right-hand side does not. So $c = 0$.
Similarly, the left hand side contains the bx^2 term, but the right-hand side does not so $b = 0$.
Since $b = 0$ & $c = 0$, the above eqth implies $a = 0$.

2.42 Spanning list of the right length is a basis

Suppose V is a finite dim vector space. Then every spanning list of vectors in V w/ length $\dim V$ is a basis of V.

continued proof for 2.41

So $1, (x-5)^2, (x-5)^3$ is lin indep.
Since the length of $1, (x-5)^2, (x-5)^3$ is 3 and $\dim U = 3$, by 2.39 $1, (x-5)^2, (x-5)^3$ is a basis of V. Note that $\dim U$ is at most 4, but it cannot equal 4 b/c if $\dim U = 4$, then we can extend a basis of U to a basis of $P_3(\mathbb{R})$, which would produce a list w/ length greater than 4. So $\dim U = 3$.

Now back to 2.42

Proof: Suppose $\dim V = n$ & $v_1, \ldots, v_n$ spans V. By 2.31 of Axler, we can reduce (if necessary) to a basis of V. But every basis of V has length n, so in this case we do not need to reduce anything, we do not need to remove any elements of $v_1, \ldots, v_n$. Therefore, $v_1, \ldots, v_n$ itself is a basis of V.

2.43 Dimension of a sum

If U_1 & U_2 are subspaces of a finite-dim. vector space V, the
$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$

Proof:

Let $m = \dim(U_1 \cap U_2)$, and let $u_1, \ldots, u_m$ be a basis of $U_1 \cap U_2$. Then it is lin indep. in U_1. So by 2.33 of Axler, we can extend this list to a basis $u_1, \ldots, u_m, v_1, \ldots, v_j$ of U_1 which means $\dim U_1 = m + j$. Similarly, $u_1, \ldots, u_m$ is lin indep in U_2. So by 2.33 of Axler, we can extend this list to a basis $u_1, \ldots, u_m, w_1, \ldots, w_k$ of U_2 which means $\dim U_2 = m + k$.

We will prove that the list $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is a basis of $U_1 + U_2$. We have $U_1, U_2 \subset \text{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$, which means $\text{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k) = U_1 + U_2$. So the dimensions of $U_1 + U_2$ & $\text{span}(u_1, \ldots, u_m, v_1, \ldots, w_1, \ldots, w_k)$ are equal. If $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is lin indep, then by 2.39 of Axler it would be a basis.

Prove that $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is lin indep.

Re explain example 2.41: Why does $U = \{p \in P_3(\mathbb{R}) : p'(5) = 0\}$ have dimension 3?

Since we proved $1, (x-5)^2, (x-5)^3$ is lin indep in U, $\dim U$ is 3 or 4

Since U is a subspace of $P_3(\mathbb{R})$, by 2.38 of Axler

$3 \le \dim U \le \dim P_3(\mathbb{R}) = 4$ (If U is a subspace of V, then $\dim U \le \dim V$)

However, $q = x - 5 \in P_3(\mathbb{R})$, BUT $q = x - 5 \in U$ b/c P

$q'(x) = 1$

$$q'(5) = 1$$
$$(q'(5) \neq 0)$$
So $q \notin U$.

We found a polynomial such as $x - 5$ that is in $P_3(\mathbb{R})$ but not in U

Therefore, $U \neq P_3 \mathbb{R} = 4$

So we conclude $\dim U = 3$

Back to 2.43

Suppose $a_1 u_1 + \ldots + a_m u_m + b_1 v_1 + \ldots + b_j v_j + c_1 w_1 + \ldots + c_k w_k = 0$

Need to prove: $a_1 = 0, \ldots, a_m = 0, b_1 = 0, \ldots, b_j = 0, c_1 = 0, \ldots, c_k =$

Since $u_1, \ldots, u_m, v_1, \ldots, v_j$ is a basis of U_1, we have

$c_1 w_1 + \ldots + c_k w_k = -a_1 u_1 - \ldots - a_m u_m - b_1 v_1 - \ldots - b_j v_j \in U_1$

Since $w_1, \ldots, w_k \in U_2$ we have $c_1 w_1 + \ldots + c_k w_k \in U_2$ So

$c_1 w_1, \ldots, c_k w_k \in U_1 \cap U_2$

Since we introduced $u_1, \dots, u_m$ to be a basis of $U_1 \cap U_2$, we can write

$$c_1 w_1 + \dots + c_k w_k = d_1 u_1 + \dots + d_m u_m$$

for some $d_1, \dots, d_m \in \mathbb{F}$. This means

$$c_1 w_1 + \dots + c_k w_k - d_1 v_1 - \dots - d_m v_m = 0$$

Since $v_1, \dots, v_m, w_1, \dots, w_k$ is lin indep, all the scalars are zero, $c_1 = 0, \dots, c_k = 0, d_1 = 0, \dots, d_m = 0$. In particular $c_1 = 0, \dots, c_k = 0$.

So the original eqtn,

$$a_1 v_1 + \dots + a_m v_m + b_1 v_1 + \dots + b_j v_j + c_1 w_1 + \dots + c_k w_k = 0$$

reduces to

$$a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_j v_j = 0$$

Since $u_1, \dots, u_m, v_1, \dots, v_j$ is a basis of U_1, it is lin indep, so $a_1 = 0, \dots, a_m = 0, b_1 = 0, \dots, b_j = 0$.

So all scalars are zero. So $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is lin indep. Then by 2.39 of Axler, it is also a basis of $U_1 + U_2$.

Therefore, we have

$$\begin{aligned}\dim(U_1 + U_2) &= m + j + k \\ &= (m + j) + (m + k) - m \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)\end{aligned}$$