LECTURE 07

07-03-19

2.28 Example

- The list $(1,-1,0),(1,0,-1)$ is a basis of

$$\{(x,y,z) \in \mathbb{F}^3 : x+y+z\ 0\}$$

$$\begin{aligned}(x,y,z) &= (x,y,-x-y)\\ &= (x,0,-x) + (0,y,-y)\\ &= x(1,0,-1) + y(0,1,-1)\end{aligned}$$

so $(1,0,-1),(0,1,-1)$ spans U

Suppose $a, b \in \mathbb{F}$ satisfy

$$a(1,0,-1) + b(0,1,-1) = (0,0,0)$$

$$(a,0,-a) + (0,b,-b) = (0,0,0)$$

$$(a,b,-a-b) = (0,0,0)$$

Equate coordinates

$$\boxed{\begin{aligned}a&=0\\ b&=0\end{aligned}}$$

$$-a-b=0$$

Therefore the list $(1,0,-1),(0,1,-1)$ is linearly independent

so $(1,0,-1),(0,1,-1)$ is a basis of U

- The list $1, z, \dots, z^m$ is a basis of $\mathcal{P}_m(\mathbb{F})$

Every polynomial $p \in \mathcal{P}_m(\mathbb{F})$ is written

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$$

for all $z \in \mathbb{F}$ for all $a_0, a_1, a_2, \dots, a_m \in \mathbb{F}$

Notice that $p(z)$ is a linear combination of $1, z, z^2, \dots, z^m$

Also for example

$1, z+3, (z+3)^2, \dots, (z+3)^m$ is a basis of $\mathcal{P}_m(\mathbb{F})$

Another example,

$1, z-6, (z6)^2$ is a basis of $\mathcal{P}_z(\mathbb{F})$

2C Dimension

we need to recall 2.23 of Axler from Section 2.B

2.23 : Length of linearly independent list $\leq$ length of spanning list in a finite-dimensional vector space.

2.35 The length of a basis of a vector space does not depend on the basis.

Any two bases of a finite dimensional vector space have the same length. (same number of vectors in the basis)

Proof: Suppose V is a finite-dimensional vector space. Let $B_1 = v_1, \dots, v_m$ and $B_2 = w_1, \dots, w_n$ be two bases of V. Then, by 2.23 of Axler, the length of B_1 is less than or equal to the length of B_2. Interchange (swap) the roles of B_1 and B_2. By 2.23 of Axler, the length of B_2 is less than or equal to the length of B_1

$B_1 = v_1, \dots, v_m$

$B_2 = w_1, \dots w_n$

$v_1, \dots v_m, w_1, \dots, w_n \in V$

In other words we have

length of $B_1 \leq$ length of B_2

and

length of $B_2 \leq$ length of B_1

Therefore

length of B_1 = length of B_2

In other words, the two bases have the same length

2.36 Definition

The dimension of a finite-dimensional vector space V is the length of any basis B of V.

The dimensions of V is denoted $\dim V$

2.37 Example

- $\dim \mathbb{F}^n = n$ because the length of any basis of $\mathbb{F}^n$ is n (any basis of $\mathbb{F}^n$ contains n elements)
- $\dim P_m(\mathbb{F}) = m+1$ because, for example, $1, z, z^2, \ldots, z^m$ is a basis of $P_m(\mathbb{F})$ and the length of the basis is $m+1$

2.38 Dimension of a subspace

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$

Proof: Since V is finite-dimensional and U is a subspace of U, by 2.26 of Axler, U is also finite-dimensional. By 2.32 of Axler, there exist a basis of U and a basis of V.

This means in particular that the basis of U is a linearly independent list in U and the basis of V is a spanning list of V.

Recall from 2.23: length of linearly independent list $\leq$ length of spanning list (of Axler)

The length of our linearly independent list in U is $\dim U$. Likewise, the length of our spanning list in V is $\dim V$.

Therefore, $\dim U \leq \dim V$

Useful result!

2.39 Linearly independent list of the right length is a basis

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V

Proof: Suppose $\dim V = n$. Let $v_1, \dots, v_n$ be a linearly independent list in V. By 2.33 of Axler, we can extend $v_1, \dots, v_n$ (if necessary) to a basis of V. But every basis of V has length n. Since $v_1, \dots, v_n$ already has length n, in this case, we do NOT need to extend to a basis of V. this means, $v_1, \dots, v_n$ is itself a basis of V.

2.40 Example

show that the list $(5,7), (4,3)$ is a basis of $\mathbb{F}^2$

Proof: We will show that $(5,7), (4,3)$ is linearly independent

Suppose $a_1, a_2 \in \mathbb{F}$ satisfy

$$a_1(5,7) + a_2(4,3) = (0,0)$$

Then we have

$$(5a_1 + 4a_2, 7a_1 + 3a_2) = (0,0)$$

Equate the coordinates to get the system of equations

$$5a_1 + 4a_2 = 0$$
$$7a_1 + 3a_2 = 0$$

System-solve to get

$$a_1 = 0 \quad a_2 = 0$$

So the list $(5,7), (4,3)$ is linearly independent in $\mathbb{F}^2$

Since $(5,7), (4,3)$ has length 2 and $\dim \mathbb{F}^2 = 2$ by 2.39 of Axler We conclude that $(5,7), (4,3)$ is a basis of $\mathbb{F}^2$

2.41 Example

Show that $1, (x-5)^2, (x-5)^3$ is a basis of the subspace U of $P_3(\mathbb{R})$ defined by

$$U = \{p \in P_3(\mathbb{R}) : p'(5) = 0\}$$

Proof: Let $p_1(x) = 1$. Then $p_1'(x) = 0$ so $p_1'(5) = 0$ and so $p_1 = 1 \in U$

Let $p_2(x) = (x-5)^2$ Then $p_2'(x) = 2(x-5)$.
so $p_2'(5) = 0$, and so $p_2(x-5)^2 \in U$
Let $p_3(x) = (x-5)^3$. Then $p_3'(x) = 3(-5)^2$
so $p_3'(5) = 0$, and so $p_3 = (x-5)^3 \in U$
Therefore, $1, (x-5)^2, (x-5)^3 \in U$

Next, suppose $a, b, c \in \mathbb{R}$ satisfy

$$a + b(x-5)^2 + c(x-5)^3 = 0$$

for all $x \in \mathbb{R}$ Notice that the left-hand side of the above equation contains the cx^3 term, but the right-hand side does not. So $c = 0$

Similarly, the left hand side contains the bx^2 term but the right hand side does not.
so $b = 0$
Since $b = 0$ and $c = 0$, the above equation implies $a = 0$.

2.42 Spanning list of the right length is a basis

Suppose V is a finite-dimensional vector space. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

So $1, (x-5)^2, (x-5)^3$ is linearly independent.
Since the length of $1, (x-5)^2, (x-5)^3$ is 3 and $\dim U = 3$ by 2.39
$1, (x-5)^2, (x-5)^3$ is a basis of V.

Note that $\dim U$ is at most 4, but it cannot equal 4 because if $\dim U = 4$ then we can extend a basis of U to a basis of $P_3(\mathbb{R})$, which would produce a list with length greater than 4. So $\dim U = 3$ (see Axler p.46)

Proof: Suppose $\dim V = n$ and $v_1, \dots, v_n$ spans V

By 2.31 of Axler, we can reduce (if necessary) to a basis of V. But every basis of V has length n, so, in this case we do not need to reduce anything. we do not need to remove any elements of $v_1, \dots, v_n$.

Therefore $v_1, \dots, v_n$ itself is a basis of V

2.43 Dimension of a sum

If U_1 and U_2 are subspaces of a finite dimensional vector space V, then

$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim (U_1 \cap U_2)$

Proof: Let $m = \dim(U_1 \cap U_2)$ and let $u_1, \dots, u_m$ be a basis of $U_1 \cap U_2$. Then it is linearly independent in U_1

So by 2.33 of Axler, we can extend this list to a basis $u_1, \dots, u_m, v_1, \dots, v_j$ of U_1, which means $\dim U_1 = m + j$

Similarly: $u_1, \dots, u_m$ is linearly independent in U_2. So, by 2.33 of Axler, we can extend this list to a basis $u_1, \dots, u_m, w_1, \dots w_k$ of U_2, which means $\dim U_2 = m + k$.

we will prove that the list $u_1, \dots, u_m, v_1, \dots v_j, w_1, \dots, w_k$ is a basis of $U_1 + U_2$

we have $U_1, U_2 \subset \operatorname{span}(u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots w_k)$ which means

$\operatorname{span}(u_1, \dots u_m, v_1, \dots, v_j, w_1, \dots w_k) = U_1 + U_2$

So the dimensions of $U_1 + U_2$ and span$(u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k)$ are equal. If $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is linearly independent, then by 2.39 of Axler it would be a basis.

Prove that $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is linearly independent

$\boxed{\text{DISCUSSION 02}}$

Re-explain Example 2.41: why does

$$U = \{p \in P_3(\mathbb{R}) : p'(5) = 0\}$$

have dimension 3

Since we proved $1, (x-5)^2, (x-5)^3$ is linearly independent in U, dim U is 3 or 4

Since U is a subspace of $P_3(\mathbb{R})$, by 2.38 of Axler

$$3 \le \dim U \le \dim P_3(\mathbb{R}) = 4$$

(If U is a subspace of V, then $\dim U \le \dim V$)

However, $x-5 \in P_3(\mathbb{R})$. But $q = x-5 \notin U$ because

$$q'(5) = 1$$
$$(q'(5) \neq 0) \qquad q'(x) = 1$$

So $q \notin U$

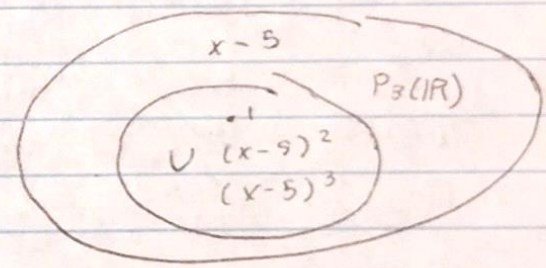
We found a polynomial such as $x-5$ that is in $P_3(\mathbb{R})$ but not in U

Therefore, $U \neq P_3(\mathbb{R})$

This means we have

$$3 \le \dim U < \dim P_3(\mathbb{R}) = 4$$

so we conclude $\dim U = 3$.



2.43 Prove $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is linearly independent

Suppose

$$a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_j v_j + c_1 w_1 + \dots + c_k w_k = 0$$

Need to prove: $a_1 = 0, \dots, a_m = 0, b_1 = 0, \dots, b_j = 0, c_1 = 0, \dots, c_k = 0$

Since $u_1, \dots, u_m, v_1, \dots, v_j$ is a basis of U_1, we have

$$c_1 w_1 + \dots + c_k w_k = -a_1 u_1 - \dots - a_m u_m - b_1 v_1 - \dots - b_j v_j \in U_1$$

Since $w_1, \dots, w_k \in U_2$ we have
$c_1 w_1 + \dots + c_k w_k \in U_2$

So
$c_1 w_1 + \dots + c_k w_k \in U_1 \cap U_2$

Since we introduced $u_1, \dots, u_m$ to be a basis of $U_1 \cap U_2$, we can write

$$c_1 w_1 + \dots + c_k w_k = d_1 v_1 + \dots + d_m u_m$$

for some $d_1, \dots, d_m \in \mathbb{F}$. This means

$$c_1 w_1 + \dots + c_k w_k - d_1 v_1 - \dots - d_m v_m = 0$$

Since $u_1, \dots, u_m, w_1, \dots w_k$ is linearly independent all the scalars are zero:

$$c_1 = 0, \dots, c_k = 0, d_1 = 0, \dots, d_m = 0$$

In particular

$$c_1 = 0, \dots, c_k = 0$$

Since the original eq

$$a_1 v_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_j v_j + c_1 w_1 + \dots + c_k w_k = 0$$

reduces to

$$a_1 v_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_j v_j = 0$$

Since $u_1, \dots, u_m, v_1, \dots, v_j$ is a basis of U_1 it is linearly independent so

$$a_1 = 0, \dots, a_m = 0, b_1 = 0, \dots, b_j = 0$$

So all scalars are zero.

So $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is linearly independent

Then by 2.39 of Axler, it is also a basis of $U_1 + U_2$. Therfore we have

$$\begin{aligned} \dim(U_1 + U_2) &= m + j + k \\ &= (m + j) + m + k) - m \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) \end{aligned}$$