2.C. Dimension

We need to recall 2.23 of Axler from section 2.B.

2.23: Length of linearly independent ~~list~~ list $\leq$ Length of spanning list in a finite-dimensional vector space.

2.35 The length of a basis of a vector space does not depend on the basis.

Any two bases of a finite-dimensional vector space have the same length. (same number of vectors in the bases).

Proof: Suppose V is a finite-dimensional vector space.

Let B_1 and B_2 be two bases of V.

$B_1 = v_1, \ldots, v_m$ $\quad$ $B_2 = w_1, \ldots, w_n$

$B_1 = v_1, \ldots, v_m$

$B_2 = w_1, \ldots, w_n$

$v_1, \ldots, v_m, w_1, \ldots, w_n \in V$.

Then, by 2.23 of Axler, the length of B_1 is less than or equal to the length of B_2. Interchange (swap) the roles of B_1 and B_2. By 2.23 of Axler, the length of B_2 is less than or equal to the length of B_1.

In other words, we have length of $B_1 \leq$ length of B_2
and length of $B_2 \leq$ length of B_1.

Therefore, length of B_1 = length of B_2.

In other words, the two bases have the same length.

2.36 Definition

The dimension of a finite-dimensional vector space V is the length of any bases B of V.

The dimension of V is denoted $\dim V$.

2.32 Example

- $\dim \mathbb{F}^n = n$ because the length of any basis of $\mathbb{F}^n$ is n (any basis of $\mathbb{F}^n$ contains n elements)
- $\dim \mathcal{P}_m(\mathbb{F}) = m+1$ because, for example, $1, z, z^2, \ldots, \boxed{z^m}$ is a basis of $\mathcal{P}_m(\mathbb{F})$ and the length of the basis is $m+1$.

2.38 Dimension of a subspace

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Proof: Since V is finite-dimensional and U is a subspace of V, by 2.26 of Axler, U is also finite-dimensional. By 2.32 of Axler, there exist a basis of U and a basis of V. This means in particular, that the basis of U is a linearly independent list in U and the basis of V is a spanning list of V.

Recall from 2.23 of Axler: length of linearly independent list $\leq$ length of spanning list. The length of our linearly independent list in U is $\dim U$. Likewise, the length of our spanning list in V is $\dim V$.

Therefore, $\dim U \leq \dim V$.

※ Useful result!

2.39 Linearly independent list of the right length is a basis

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Proof: Suppose $\dim V = n$. Let $v_1, \ldots, v_n$ be a linearly independent list in V. By 2.33 of Axler, we can extend $v_1, \ldots, v_n$ $\boxed{\text{(if necessary)}}$ to a basis of V. But every basis of V has length n. Since $v_1, \ldots, v_n$ already has length n, in this case, we do NOT need to extend to a basis of V. This means, $v_1, \ldots, v_n$ is itself a basis of V.

2.40 Example

Show that the list $(5,7), (4,3)$ is a basis of $\mathbb{F}^2$.

Proof: We will show that $(5,7), (4,3)$ is linearly independent.

Suppose $a_1, a_2 \in \mathbb{F}$ satisfy $a_1(5,7)+a_2(4,3) = (0,0)$

Then we have $(5a_1+4a_2, 7a_1+3a_2)=(0,0)$

Equate the coordinates to get the system of equations

$$5a_1+4a_2=0$$
$$7a_1+3a_2=0$$

System-solve to get $a_1=0$, $a_2=0$.

So the list $(5,7)(4,3)$ is linearly ~~dep~~ independent in $\mathbb{F}^2$.

Since $(5,7), (4,3)$ has length 2 and $\dim \mathbb{F}^2=2$, by 2.39 of Axler, we conclude that $(5,7), (4,3)$, is a basis of $\mathbb{F}^2$.

2.41 Example

Show that $1, (x-5)^2, (x-5)^3$ is a basis of the subspace U of $\mathcal{P}_3(\mathbb{R})$ defined by $U = \{p \in \mathcal{P}_3(\mathbb{R}) : p'(5)=0\}$.

Proof: Let $p_1(x)=1$. Then $p_1'(x)=0$, so $p_1'(5)=0$.

and so $p_1 = 1 \in U$.

Let $p_2(x)=(x-5)^2$. The $p_2'(x)=2(x-5)$.

So $p_2'(5)=0$, and so $p_2=(x-5)^2 \in U$.

Let $p_3(x)=(x-5)^3$. Then $p_3'(x)=3(x-5)^2$.

So $p_3'(5)=0$, and so $p_3=(x-5)^3 \in U$.

Therefore, $1, (x-5)^2, (x-5)^3 \in U$.

Next, suppose $a,b,c \in \mathbb{R}$ satisfy $a+b(x-5)^2+c(x-5)^3=0$. for all $x \in \mathbb{R}$.
Notice that the left-hand side of the above equation contains the cx^3 term, but the right-hand side does not. So $c=0$.
Similarly, the left-hand side contains the bx^2 term, but the right-hand side does not. So $b=0$

Since $b=0$ and $c=0$, the above equation implies $a=0$.

So, $1, (x-5)^2, (x-5)^3$ is linearly independent.

Since the length of $1, (x-5)^2, (x-5)^3$ is 3 and $\dim U=3$, by 2.39, $1, (x-5)^2, (x-5)^3$ is a basis of V.

Note that $\dim U$ is at most 4, but it cannot equal 4 because if $\dim U=4$, then we can extend a basis of U to a basis of $P_3(\mathbb{R})$, which would produce a list with length greater than 4. So $\dim U=3$.

(See Axler. P.46).

Re-explain Example 2.41 why does $U=\{p\in P_3(\mathbb{R}) : p'(5)=0\}$ have dimension 3?

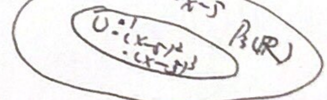
Since we proved $1, (x-5)^2, (x-5)^3$ is linearly independent in U, $\dim U$ is 3 or 4. Since U is a subspace of $P_3(\mathbb{R})$, by 2.38 of Axler, $3 \le \dim U \le \dim P_3(\mathbb{R}) = 4$, (If U is a subspace of V, then $\dim U \le \dim V$). However, $q=x-5\in P_3(\mathbb{R})$, BUT $q_2=x-5 \notin U$ because $q_2'(x)=1$ $q_2'(5)=1$ ($q_2'(5)\neq 0$) So $q_2 \notin U$.

We found a polynomial such as $x-5$ that is in $P_3(\mathbb{R})$ but not in U.

Therefore, $U \neq P_3(\mathbb{R})$

This means we have $3 \le \dim U < \dim P_3(\mathbb{R}) = 4$

So we conclude $\dim U = 3$.



2.42 Spanning list of the right length is a basis.

Suppose V is a finite-dimensional vector space. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

Proof: Suppose $\dim V = n$ and $v_1, \dots, v_n$ spans V.

By 2.31 of Axler, we can reduce (if necessary) to a basis of V.

But every basis of V has length n, so in this case, we do not need to reduce anything; we do not need to remove any elements of $v_1, \dots, v_n$.

Therefore, $v_1, \dots, v_n$ itself is a basis of V.

2.43 Dimension of a sum

If U_1 and U_2 are subspaces of a finite-dimensional vector space V, then,

$$\dim(U_1+U_2) = \dim U_1 + \dim U_2 - \dim(U_1\cap U_2).$$

Proof: Let $m=\dim(U_1\cap U_2)$, and let $u_1, \dots, u_m$ be a basis of $U_1\cap U_2$. Then it is linearly independent in U_1. So, by 2.33 of Axler, we can extend this list to a basis $u_1, \dots, u_m, v_1, \dots, v_j$ of U_1, which means $\dim U_1 = m+j$.

Similarly, $u_1, \ldots, u_m$ is linearly independent in U_2. So, by 2.33 of Axler, we can extend this list to a basis $u_1, \ldots, u_m, w_1, \ldots, w_k$ of U_2, which means $\dim U_2 = m+k$.

We will prove that the list $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is a basis of $U_1 + U_2$. We have $U_1, U_2 \subset \operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$, which means $\operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k) = U_1 + U_2$.

So the dimensions of $U_1 + U_2$ and $\operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$ are equal.

If $u_1, \ldots, u_m, v_1, \ldots v_j, w_1, \ldots, w_k$ is linearly independent, then by 2.39 of Axler it would be a basis.

<u>Prove that $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is linearly independent</u>

Suppose $a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_jv_j + c_1w_1 + \cdots + c_kw_k = 0$

Need to prove: $a_1 = 0, \ldots, a_m = 0, b_1 = 0, \ldots, b_j = 0, c_1 = 0, \ldots, c_k = 0$.

Since $u_1, \ldots, u_m, v_1, \ldots, v_j$ is a basis of U_1, we have

$$c_1w_1 + \cdots + c_kw_k = -a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j \in U_1$$

Since $w_1, \ldots, w_k \in U_2$, we have $c_1w_1 + \cdots + c_kw_k \in U_2$.

So $c_1w_1 + \cdots + c_kw_k \in U_1 \cap U_2$.

Since we introduced $u_1, \ldots, u_m$ to be a basis of $U_1 \cap U_2$,
we can write $c_1w_1 + \cdots + c_kw_k = d_1u_1 + \cdots + d_mu_m$
for some $d_1, \ldots, d_m \in \mathbb{F}$. This means $c_1w_1 + \cdots + c_kw_k - d_1u_1 - \cdots - d_mu_m = 0$.

Since $u_1, \ldots, u_m, w_1, \ldots, w_k$ is linearly independent, all the scalars are zero:

$$c_1 = 0, \ldots, c_k = 0, d_1 = 0, \ldots, d_m = 0.$$

In particular, $c_1 = 0, \ldots, c_k = 0$.

Similarly, $u_1, \ldots, u_m$ is linearly independent in U_2. So, by 2.33 of Axler, we can extend this list to a basis $u_1, \ldots, u_m, w_1, \ldots, w_k$ of U_2, which means $\dim U_2 = m+k$.

We will prove that the list $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is a basis of $U_1 + U_2$. We have $U_1, U_2 \subset \operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$, which means $\operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k) = U_1 + U_2$.

So the dimensions of $U_1 + U_2$ and $\operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$ are equal.

If $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is linearly independent, then by 2.39 of Axler it would be a basis.

Prove that $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is linearly independent.

Suppose $a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j + c_1 w_1 + \cdots + c_k w_k = 0$

Need to prove: $a_1 = 0, \ldots, a_m = 0, b_1 = 0, \ldots, b_j = 0, c_1 = 0, \ldots, c_k = 0$.

Since $u_1, \ldots, u_m, v_1, \ldots, v_j$ is a basis of U_1, we have

$$c_1 w_1 + \cdots + c_k w_k = -a_1 u_1 - \cdots - a_m u_m - b_1 v_1 - \cdots - b_j v_j \in U_1$$

Since $w_1, \ldots, w_k \in U_2$, we have $c_1 w_1 + \cdots + c_k w_k \in U_2$.

So $c_1 w_1 + \cdots + c_k w_k \in U_1 \cap U_2$.

Since we introduced $u_1, \ldots, u_m$ to be a basis of $U_1 \cap U_2$, we can write $c_1 w_1 + \cdots + c_k w_k = d_1 u_1 + \cdots + d_m u_m$ for some $d_1, \ldots, d_m \in \mathbb{F}$. This means $c_1 w_1 + \cdots + c_k w_k - d_1 u_1 - \cdots - d_m u_m = 0$.

Since $u_1, \ldots, u_m, w_1, \ldots, w_k$ is linearly independent, all the scalars are zero:

$$c_1 = 0, \ldots, c_k = 0, d_1 = 0, \ldots, d_m = 0.$$

In particular, $c_1 = 0, \ldots, c_k = 0$.

~~2~~ So the original equation

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j + c_1 w_1 + \cdots + c_k w_k = 0$$

reduces to $a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j = 0$

Since $u_1, \ldots, u_m, v_1, \ldots, v_j \ldots$ is a basis of U_1, it is linearly independent,

So $a_1 = 0, \ldots, a_m = 0, b_1 = 0, \ldots, b_j = 0$.

So all scalars are zero.

So $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is linearly independent.

Then by 2.39 of Axler, it is also a basis of $U_1 + U_2$.

Therefore, we have

$$\begin{aligned} \dim(U_1 + U_2) &= m + j + k \\ &= (m+j) + (m+k) - m \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) \end{aligned}$$

3.A The Vector Space of Linear Maps

3.2 Definition

A linear map from V to W is a function $T: V \to W$ that satisfies:

- additivity: $T(u+v) = Tu + Tv$ for all $u, v \in V$
- homogeneity: $T(\lambda v) = \lambda Tv$ for all $\lambda \in \mathbb{F}$ and for all $v \in V$.

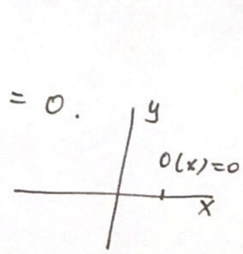
The set of all linear maps $T: V \to W$ is denoted $\mathcal{L}(V, W)$.

3.4 Examples:

(1) Zero map: Let $0 \in \mathcal{L}(V, W)$ be the zero map. defined by $0v = 0$.
$0(v)$

$0 \in \mathcal{L}(V, W)$

this is, 0 is a linear map because:



- additivity: If $u, v \in V$, then
$$\begin{aligned} 0(u+v) &= 0 \\ &= 0 + 0 \\ &= 0u + 0v \end{aligned}$$
- homogeneity: If $u \in U$ and $\lambda \in \mathbb{F}$, then
$$\begin{aligned} 0(\lambda u) &= 0 \\ &= \lambda \cdot 0 \\ &= \lambda 0(u). \end{aligned}$$