7/8 3.A The vector space of linear Maps:

3.2 Def

linear map - from V to W is a function $T: V \to W$ that satisfies:

- additivity:

  $T(u+v) = Tu + Tv$ for all $u, v \in V$

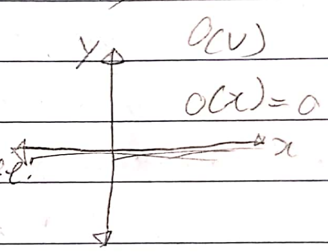
- homogeneity:

  $T(\lambda v) = \lambda T v$ for all $\lambda \in \mathbb{F}$ & for all $v \in V$

The set of all linear maps $T: V \to W$ is denoted $\mathcal{L}(V, W)$

3.4 example:

let 0 be the zero map defined by $0v = 0$



$0 \in \mathcal{L}(V, W)$

that is, 0 is a linear map because:

additivity: If $u, v \in V$, then

$0(u+v) = 0$

$= 0 + 0 = 0u + 0v$

homogeneity: If $u \in V$ & $\lambda \in \mathbb{F}$, then

$0(\lambda u) = 0$

$= \lambda \cdot 0 = \lambda 0(u)$

- Identity maps

define $I: V \to V$ by $Iv = v$

then $I \in \mathcal{L}(V, V)$ because:

additivity: If $u, v \in V$, then

$I(u+v) = u + v$

$= Iu + Iv$

homogeneity: If $\lambda \in \mathbb{F}$ & $v \in V$, then

$I(\lambda v) = \lambda v = \lambda I v.$

• Differentiation
Define $D : P(\mathbb{R}) \to P(\mathbb{R})$ by
$Dp = p'$
Then $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ because:
additivity. If $p, q \in P(\mathbb{R})$, then
$D(p+q) = (p+q)'$
$= p' + q'$
$= Dp + Dq$
homogeneity: If $\lambda \in \mathbb{F}$ and $p \in P(\mathbb{R})$, then
$D(\lambda p) = (\lambda p)' = \lambda p' = \lambda Dp$

• Integration: Define $T : P(\mathbb{R}) \to \mathbb{R}$ by
$Tp = \int_0^1 p(x)\,dx$
Then we have $T \in \mathcal{L}(P(\mathbb{R}), \mathbb{R})$ because:
additivity: If $p, q \in P(\mathbb{R})$, then
$T(p+q) = \int_0^1 (p+q)(x)\,dx$
$= \int_0^1 p(x) + q(x)\,dx$
$= \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx$
$= Tp + Tq$
homogeneity: If $\lambda \in \mathbb{F}$ and $p \in P(\mathbb{R})$, then
$T(\lambda p) = \int_0^1 (\lambda p)(x)\,dx = \int_0^1 \lambda p(x)\,dx = \lambda \int_0^1 p(x)\,dx$
$= \lambda Tp.$

Multiplication by x^2
Define $T : P(\mathbb{R}) \to P(\mathbb{R})$ by
$(Tp)(x) = x^2 p(x)$,
$\forall x \in \mathbb{R}$. Then $T \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ because:
Additivity: If $p, q \in P(\mathbb{R})$, then $\forall x \in \mathbb{R}$, we
have $(T(p+q))(x) = x^2(p+q)(x) = x^2(p(x) + q(x))$
$= x^2 p(x) + x^2 q(x) = (Tp)(x) + (Tq)(x) = ((Tp) + (Tq))x.$
So $T(p+q) = Tp + Tq$

Homogeneity: If $\lambda \in \mathbb{F}$ and $p \in \mathcal{P}(\mathbb{R})$, then, $\forall x \in \mathbb{R}$,
$(T(\lambda p))(x) = x^2(\lambda p)(x) = x^2(\lambda p(x)) = \lambda x^2 p(x)$
$= \lambda (Tp)(x)$

From $\mathbb{R}^3$ to $\mathbb{R}^2$: Define $T: \mathbb{R}^3 \to \mathbb{R}^2$ by
$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z)$
Then $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ because:
Additivity: If $(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3$, then
$T((x, y, z) + (\tilde{x}, \tilde{y}, \tilde{z})) = T(x + \tilde{x}, y + \tilde{y}, z + \tilde{z})$
$= (2(x+\tilde{x}) - (y+\tilde{y}) + 3(z+\tilde{z}), 7(x+\tilde{x}) + 5(y+\tilde{y}) - 6(z+\tilde{z}))$
$= (2x - y + 3z, 7x + 5y - 6z) + (2\tilde{x} - \tilde{y} + 3\tilde{z}, 7\tilde{x} + 5\tilde{y} - 6\tilde{z})$
$= T(x, y, z) + T(\tilde{x}, \tilde{y}, \tilde{z})$
Homogeneity:
If $\lambda \in \mathbb{F}$ and $(x, y, z) \in \mathbb{R}^3$, then
$T(\lambda(x, y, z)) = T(\lambda x, \lambda y, \lambda z)$
$= (2(\lambda x) - \lambda y) + 3(\lambda z), \lambda(\lambda x) + 5(\lambda y) - 6(\lambda z))$
$= (\lambda(2x - y + 3z), \lambda(7x + 5y - 6z))$
$= \lambda(2x - y + 3z, 7x + 5y - 6z)$
$= \lambda T(x, y, z)$.

3.5 linear Maps & basis of domain
Suppose $v_1, \dots, v_n$ is a basis of V & $w_1, \dots, w_n \in W$
There $\exists$ a unique linear map $T: V \to W$
such that $Tv_j = w_j$
for each $j = 1, \dots, n$
Proof: 1st prove linear map T exists:
define $T: V \to W$ by
$T(c_1 v_1 + \dots + c_n v_n) = c_1 w_1 + \dots + c_n w_n$
for some $c_1, \dots, c_n \in \mathbb{F}$. Since $v_1, \dots v_n$ is a basis,
every vector in V is of form $c_1 v_1 + \dots + c_n v_n$.
So map T is defined a function $T: V \to W$
For each $j = 1, \dots, n$, if $c_j = \{ {1 \text{ if } j = i \atop 0 \text{ if } j \neq i}$ otherwise,
then T satisfies $Tv_j = w_j$.

Next prove T is linear. ($T \in \mathcal{L}(V,W)$)

• Additivity: If $u, v \in V$ then since $v_1, \dots, v_n$ is a basis of V, we can write:

$$u = a_1 v_1 + \dots + a_n v_n$$
$$\& \quad v = c_1 v_1 + \dots + c_n v_n$$

for some $a_1, \dots, a_n, c_1, \dots, c_n \in \mathbb{F}$

$$\begin{aligned} T(u+v) &= T(a_1 v_1 + \dots + a_n v_n) + (c_1 v_1 + \dots + c_n v_n)) \\ &= T((a_1 + c_1) v_1 + \dots + (a_n + c_n) v_n) \\ &= (a_1 + c_1) w_1 + \dots + (a_n + c_n) w_n \\ &= (a_1 w_1 + \dots a_n w_n) + (c_1 w_1 + \dots + c_n w_n) \\ &= T(a_1 v_1 + \dots a_n v_n) + T(c_1 v_1 + \dots + c_n v_n) \\ &= Tu + Tv \end{aligned}$$

• Homogeneity: If $\lambda \in \mathbb{F}$ & $v \in V$, then we can write,

$$v = c_1 v_1 + \dots c_n v_n$$

for some $c_1, \dots, c_n \in \mathbb{F}$ so we have

$$\begin{aligned} T(\lambda v) &= T(\lambda(c_1 v_1 + \dots + c_n v_n)) \\ &= T((\lambda c_1) v_1 + \dots + (\lambda c_n) v_n) \\ &= \lambda c_1 w_1 + \dots + \lambda c_n w_n \\ &= \lambda (c_1 w_1 + \dots c_n w_n) \\ &= \lambda T(c_1 v_1 + \dots c_n v_n) = \lambda T v \end{aligned}$$

Therefore, $T: V \to W$ satisfies additivity & homogeneity. So $T \in \mathcal{L}(V,W)$.

Finally prove T is unique.

Suppose $T \in \mathcal{L}(V,W)$ & T satisfies $Tv_j = w_j \ \exists\ j = 1, \dots, n$.

Let $c_1, \dots, c_n \in \mathbb{F}$, then the additivity & homogeneity of T gives us:

$$T(c_1 v_1 + \dots c_n v_n) \underset{\text{additivity}}{=} T(c_1 v_1) + \dots + T(c_n v_n) \underset{\text{homogeneity}}{=} c_1 T v_1 + \dots + c_n T v_n = c_1 w_1 + \dots + c_n w_n$$

So T is uniquely determined on $\mathrm{span}(v_1, \dots, v_n)$

But $v_1, \dots, v_n$ is a basis of V, $\mathrm{span}(v_1, \dots, v_n) = V$

So T is uniquely determined on V ∎

Addition and scalar multiplication on $\mathcal{L}(V, W)$

3.6 $\underline{\text{Idef}}$

Suppose $S, T \in \mathcal{L}(V, W)$ & $\lambda \in \mathbb{F}$

The sum $S+T$ is a linear map defined by

$(S+T)(v) = Sv + Tv$

The product,

$(\lambda T)(v) = \lambda(Tv)$

3.7 $\mathcal{L}(V, W)$ is a vector space

The set $\mathcal{L}(V, W)$ is a vector space with respect to operations defined in Idef 3.6.

Proof: Let $R, S, T \in \mathcal{L}(V, W)$ and $a, b \in \mathbb{F}$ be arbitrary

Commutativity

$\forall v \in V$, we have

$(S+T)v = Sv + Tv = Tv + Sv = (T+S)v$

So $S+T = T+S$.

Associativity

$\forall v \in V$, we have

$((R+S)+T)v = (R+S)v + Tv = Rv + Sv + Tv$

$= Rv + (S+T)v = (R+(S+T))v$

So $(R+S)+T = R+(S+T)$

Additive Identity

Let $0 \in \mathcal{L}(V, W)$ be the zero function $\forall v \in V$.

$(T+0)v = Tv = 0v = Tv + 0 = Tv$ so $T+0 = T$

Additive inverse

Note that we have $-T \in \mathcal{L}(V, W)$ $\forall v \in V$,

$T(+(-T))v = Tv + (-T)v = Tv - Tv = 0 = 0v$

So $T+(-T) = 0$

Multiplicative identity:

$\forall v \in V$

$(1T)v = 1Tv = Tv$

So $1T = T$

Distributive properties

$\forall a \in F$ $\forall v \in V$, we have

$$(a(S+T))v = a(Sv+Tv) = aSv + aTv = (aS+aT)v$$

So $a(S+T) = aS + aT$

$\forall a, b \in F$ $\forall v \in V$, we have $((a+b)T)v = (a+b)Tv$

$= aTv + bTv = (aT+bT)v$.

So $(a+b)T = aT + bT$

Therefore, $\mathcal{L}(V,W)$ is a vector space with respect to defined operations.

3.8 Def

If $S \in \mathcal{L}(V,W)$, $T \in \mathcal{L}(U,V)$, the the product is:

$$(ST)u = S(Tu) \quad \forall u \in U$$

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & V \arrow[r, "S"] & W \end{tikzcd}$$

3.9 Algebraic prop. of products of linear maps

Associativity:

If R, S, T are linear maps so that product RST works, then $(RS)T = R(ST)$

Identity:

If $T \in \mathcal{L}(V,W)$ & $I : V \to V$ is an identity map, then $TI = T = IT$

Distributive prop:

If $T, T_1, T_2 \in \mathcal{L}(U,V)$ & $S, S_1, S_2 \in \mathcal{L}(V,W)$, then

$(S_1 + S_2)T = S_1T + S_2T$ & $S(T_1+T_2) = ST_1 + ST_2$

3.11 linear maps take 0 to 0

If $T \in \mathcal{L}(V, W)$, then $T(0) = 0$

Proof: Since T is linear, we can use additivity to get

$$T(0) = T(0+0)$$

additivity $= T(0) + T(0) = 2T(0)$

Therefore $T(0) = 0$, as defined. ∎