July. 08 Monday.

3A. The vector space of linear maps.

3.2 Definition

A linear map from V to W is a function $T: V \to W$ that satisfies:

- additivity.

  $T(u+v) = T(u) + T(v)$ for all $u, v \in V$

- homogeneity.

  $T(\lambda v) = \lambda T(v)$ for all $v \in V$, $\lambda \in \mathbb{F}$

The set of all linear maps $T: V \to W$ is denoted $\mathcal{L}(V, W)$

3.4 Examples

Let 0 be the zero map defined by $0v = 0$.

$0 \in \mathcal{L}(V, W)$

that is, 0 is a linear map because:

- additivity: If $u, v \in V$, then

$$\begin{aligned} 0(u+v) &= 0 \\ &= 0 + 0 \\ &= 0u + 0v \end{aligned}$$

- homogeneity: If $u \in V$ and $\lambda \in \mathbb{F}$, then

$$\begin{aligned} 0(\lambda u) &= 0 \\ &= \lambda 0 \end{aligned}$$

$= \lambda T(u)$

☆ · identity map:

Define $I: V \to V$ by $Iv = v$

Then $I \in \mathcal{L}(V, V)$ because

· additivity: If $u, v \in V$, then

$$I(u+v) = u+v$$
$$= I(u) + Iv$$

· homogeneity: If $\lambda \in \mathbb{F}$ and $v \in V$, then

$$I(\lambda v) = \lambda v$$
$$= \lambda I(v)$$

☆ · Differentiation:

Define $D: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by

$$Dp = p'.$$

Then $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ because:

additivity: If $p, q \in \mathcal{P}(\mathbb{R})$, then

$$D(p+q) = (p+q)'$$
$$= p' + q'$$
$$= Dp + Dq$$

homogeneity: If $\lambda \in \mathbb{F}$, $p \in \mathcal{P}(\mathbb{R})$, then

$$D(\lambda p) = (\lambda p)'$$
$$= \lambda D(p)$$

☆ · Integration:

Define $T: \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ by

$$Tp = \int_0^1 p(x)\, dx$$

Then we have $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ because:

- Additivity: If $p, q \in \mathcal{P}(\mathbb{R})$, then

$$T(p+q) = \int_0^1 (p+q)(x)\,dx$$
$$= \int_0^1 p(x) + q(x)\,dx$$
$$= \int_0^1 p(x)dx + \int_0^1 q(x)\,dx$$
$$= Tp + Tq$$

- Homogeneity: If $\lambda \in \mathbb{F}$ and $p \in \mathcal{P}(\mathbb{R})$, then

$$T(\lambda p) = \int_0^1 (\lambda p)(x)\,dx$$
$$= \int_0^1 \lambda p(x)\,dx$$
$$= \lambda \int_0^1 p(x)\,dx$$
$$= \lambda Tp$$

- Multiplication by x^2

Define: $T: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x)$ for all $x \in \mathbb{R}$. Then $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ because:

- Additivity: If $p, q \in \mathcal{P}(\mathbb{R})$, then for all $x \in \mathbb{R}$, we have

$$(T(p+q))(x) = x^2 (p+q)(x)$$
$$= x^2 (p(x) + q(x))$$
$$= x^2 p(x) + x^2 q(x)$$
$$= (Tp)(x) + (Tq)(x)$$
$$= ((Tp) + (Tq))(x)$$

$\therefore$ $T(p+q) = Tp + Tq$.

- Homogeneity: If $\lambda \in \mathbb{F}$, and $p \in \mathcal{P}(\mathbb{R})$, then for all $x \in \mathbb{R}$, we have.

$$(T(\lambda p))(x) = x^2 (\lambda p)(x)$$
$$= x^2 (\lambda p(x))$$
$$= \lambda x^2 p(x)$$
$$= \lambda (Tp)(x)$$

☆ · From $\mathbb{R}^3$ to $\mathbb{R}^2$: Define $T: \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - y + 3z,\ 7x + 5y - 6z).$$

Then $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ because:

- additivity: If $(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3$, then

$$\begin{aligned} T((x,y,z)+(\tilde{x},\tilde{y},\tilde{z})) &= T(x+\tilde{x},\ y+\tilde{y},\ z+\tilde{z}) \\ &= (2(x+\tilde{x}) - (y+\tilde{y}) + 3(z+\tilde{z}),\ 7(x+\tilde{x}) + 5(y+\tilde{y}) - 6(z+\tilde{z})) \\ &= ((2x-y+3z)+(2\tilde{x}-\tilde{y}+3\tilde{z}),\ (7x+5y-6z)+(7\tilde{x}+5\tilde{y} \\ &\quad -6\tilde{z})) \\ &= T(x,y,z) + T(\tilde{x},\tilde{y},\tilde{z}) \end{aligned}$$

- homogeneity: If $\lambda \in \mathbb{F}$ and $(x, y, z) \in \mathbb{R}^3$, then

$$\begin{aligned} T(\lambda(x,y,z)) &= T(\lambda x, \lambda y, \lambda z) \\ &= \end{aligned}$$

3.5 Linear maps and basis of domain

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$.
Then there exists a unique linear map $T: V \to W$ s.t.

$$Tv_j = w_j \quad \text{for each } j = 1, \ldots, n$$

Proof:

First, we will prove that the linear map T exists.

Define $T: V \to W$ by

$$T(c_1 v_1 + \ldots + c_n v_n) = c_1 w_1 + \ldots + c_n w_n.$$

for some $c_1, \ldots, c_n \in \mathbb{F}$, Since $v_1, \ldots, v_n$ is a basis of V. every vector in V is uniquely of the form $c_1v_1 + \cdots + c_nv_n$. So this map T as we defined above indeed define a function $T: V \to W$.

Furthermore, for each $j = 1, \ldots, m$. if

$$c_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

then, T satisfies $Tv_j = w_j$.

Next, we will prove that T is linear, that is, $T \in \mathcal{L}(V, W)$.

- additivity: If $u, v \in V$. then since $v_1, \ldots, v_n$ is a basis of V, we can write

$u = a_1v_1 + \cdots + a_nv_n$ and

$v = c_1v_1 + \cdots + c_nv_n$

for some $a_1, \ldots, a_n, c_1, \ldots, c_n \in \mathbb{F}$, So we have

$$\begin{aligned} T(u+v) &= T((a_1v_1 + \cdots + a_nv_n) + (c_1v_1 + \cdots + c_nv_n)) \\ &= T((a_1+c_1)v_1 + \cdots + (a_n+c_n)v_n) \\ &= (a_1+c_1)w_1 + \cdots + (a_n+c_n)w_n \\ &= (a_1w_1 + c_1w_1) + \cdots + (a_nw_n + c_nw_n) \\ &= (a_1w_1 + \cdots + a_nw_n) + (c_1w_1 + \cdots + c_nw_n) \\ &= T(a_1v_1 + \cdots + a_nv_n) + T(c_1v_1 + \cdots + c_nv_n) \\ &= Tu + Tv. \end{aligned}$$

- Homogeneity: If $\lambda \in \mathbb{F}$ and $v \in V$, then we can write

$v = c_1v_1 + \cdots + c_nv_n$

for some $c_1, \ldots, c_n \in \mathbb{F}$. So we have

$$
\begin{aligned}
T(\lambda v) &= T(\lambda(c_1v_1+\dots+c_nv_n)) \\
&= T((\lambda c_1)v_1+\dots+(\lambda c_n)v_n) \\
&= (\lambda c_1)w_1+\dots+(\lambda c_n)w_n \\
&= \lambda c_1w_1+\dots+\lambda c_nw_n \\
&= \lambda(c_1w_1+\dots+c_nw_n) \\
&= \lambda T(c_1v_1+\dots+c_nv_n) \\
&= \lambda Tv
\end{aligned}
$$

$\therefore$ $T: V \to W$ satisfies additivity and homogeneity

$\therefore$ $T \in \mathcal{L}(V,W)$

Finally, we need to prove that T is unique.

Suppose we have $T \in \mathcal{L}(V,W)$ and T satisfies $Tv_j = w_j$ for each $j=1,\dots,n$.

Let $c_1,\dots,c_n \in \mathbb{F}$, then the homogeneity and additivity of T give us:

$$
\begin{aligned}
T(c_1v_1+\dots+c_nv_n) &= T(c_1v_1)+\dots+T(c_nv_n) \\
&= c_1Tv_1+\dots+c_nTv_n \\
&= c_1w_1+\dots+c_nw_n
\end{aligned}
$$

So T is uniquely determined on $\operatorname{span}(v_1,\dots,v_n)$

But $v_1,\dots,v_n$ is a basis of V, meaning we have $\operatorname{span}(v_1,\dots,v_n)=V$

So T is uniquely determined on V.

Addition and scalar multiplication on $\mathcal{L}(V,W)$

3.6 Definition

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$

- The sum $S+T$ is a linear map defined by $(S+T)(v) = Sv + Tv$.
- The product λT is a linear map defined by $(\lambda T)(v) = \lambda(Tv)$

3.7 $\mathcal{L}(V, W)$ is a vector space

The set $\mathcal{L}(V, W)$ is a vector space with respect to the operations defined in Definition 3.6 of Axler.

Proof: Let $R, S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$ be arbitrary.

- Communitivity:

  For all $v \in V$, we have

  $$\begin{aligned}(S+T)v &= Sv + Tv \\ &= Tv + Sv \\ &= (T+S)v\end{aligned}$$

  $\therefore S+T = T+S$

- Associativity:

  For all $v \in V$, we have:

  $$\begin{aligned}((R+S)+T)v &= (R+S)v + Tv \\ &= Rv + Sv + Tv \\ &= Rv + (S+T)v \\ &= (R+(S+T))v\end{aligned}$$

  $\therefore (R+S)+T = R+(S+T)$

- Additive identity:

  Let $0 \in \mathcal{L}(V, W)$ be the zero function. For all $v \in V$, we have. $(T+0)v = Tv + 0v$

  $$= Tv + 0$$

$= Tv$

so $T+0=T$

- Additive inverse:

Note that we have $-T \in \mathcal{L}(V,W)$. For all $v \in V$, we have

$$
\begin{aligned}
(T+(-T))v &= Tv+(-T)v \\
&= Tv - Tv \\
&= 0 \\
&= 0v
\end{aligned}
$$

$\therefore T+(-T)=0$

- Multiplicative identity:

For all $v \in V$, we have:

$$
\begin{aligned}
(1T)v &= 1Tv \\
&= Tv
\end{aligned}
$$

$\therefore 1T=T$

- Distributive properties:

For all $a \in \mathbb{F}$, all $v \in V$, we have:

$$
\begin{aligned}
(a(S+T))v &= a(Sv+Tv) \\
&= aSv + aTv \\
&= (aS+aT)v
\end{aligned}
$$

$\therefore a(S+T) = aS+aT$

For all $a,b \in \mathbb{F}$, for all $v \in V$, we have

$$
\begin{aligned}
((a+b)T)v &= (a+b)Tv \\
&= aTv + bTv \\
&= (aT+bT)v.
\end{aligned}
$$

$\therefore (a+b)T = aT+bT$

Therefore, $\mathcal{L}(V,W)$ is a vector space with respect to the defined operations.

3.8 Definition

If $S \in \mathcal{L}(V,W)$, $T \in \mathcal{L}(V,W)$, then the product $ST \in \mathcal{L}(V,W)$ is defined by $(ST)u = S(Tu)$ for all $u \in V$.

3.9 Algebraic properties of products of linear maps.

- Associativity

  If R, S, T are linear maps such that the product RST makes sense. then $(RS)T = R(ST)$
- Identity

  If $T \in \mathcal{L}(V,W)$ and $I: V \to V$ is an identity map, then $T \cdot I = T = IT$
- Distributive properties

  If $T, T_1, T_2 \in \mathcal{L}(U,V)$ and $S, S_1, S_2 \in \mathcal{L}(V,W)$, then $(S_1+S_2)T = S_1T + S_2T$ and

  $S(T_1+T_2) = ST_1 + ST_2$

3.11 Linear maps take 0 to 0.

If $T \in \mathcal{L}(V,W)$, then $T(0) = 0$.

Proof: Since T is linear, we can use addition to get

$$\begin{aligned} T(0) &= T(0+0) \\ &= T(0) + T(0) \\ &= 2T(0) \end{aligned}$$

$\therefore$ $T(0) = 0$, as desired.