7/8 Section 3.A. The vector space of Linear Maps

3.2 Definition:

A linear map from V to W is a function $T: V \to W$ that satisfies:

- additivity

  $T(u+v) = Tu + Tv$ for all $u, v \in V$

- homogeneity

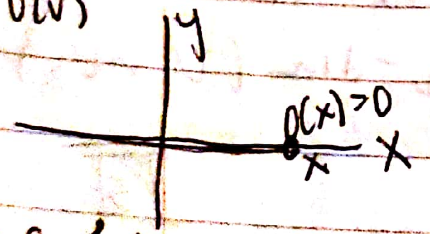
  $T(\lambda v) = \lambda Tv$ for all $\lambda \in \mathbb{F}$ and for all $v \in V$.

The set of all linear maps $T: V \to W$ is denoted $\mathcal{L}(V, W)$.

3.4 Examples: | zero map

Let 0 be the zero map $\in \mathcal{L}(V,W)$ defined by $0v = 0$.

$0(v)$



$0 \in \mathcal{L}(V, W)$ that is, 0 is a linear map because:

additivity: If $u, v \in V$, then $0(u+v) = 0$
$= 0 + 0$
$= 0u + 0v$

homogeneity: If $u \in V$ and $\lambda \in \mathbb{F}$, then $0(\lambda u) = 0$
$= \lambda \cdot 0$
$= \lambda 0(u)$.

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- **Identity map**

Define $I : V \to V$ by $Iv = v$.

Then $I \in \mathcal{L}(V, V)$ because:

Additivity: If $u, v \in V$, then $I(u+v) = u+v$
$= Iu + Iv$

Homogeneity: If $\lambda \in \mathbb{F}$ and $v \in V$, then
$I(\lambda v) = \lambda v$
$= \lambda Iv.$

- **Differentiation**

Define $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by
$Dp = p'$.

Then $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ because:

Additivity: If $p, q \in \mathcal{P}(\mathbb{R})$, then
$D(p+q) = (p+q)'$
$= p' + q'$
$= Dp + Dq$

Homogeneity: If $\lambda \in \mathbb{F}$ and $p \in \mathcal{P}(\mathbb{R})$ then
$D(\lambda p) = (\lambda p)'$
$= \lambda p'$
$= \lambda Dp$

• Integration: Define $T: \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ by

$$Tp = \int_0^1 p(x)\,dx$$

Then we have $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ because:

Additivity: If $p, q \in \mathcal{P}(\mathbb{R})$, then

$$T(p+q) = \int_0^1 (p+q)(x)\,dx$$
$$= \int_0^1 p(x) + q(x)\,dx$$
$$= \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx$$
$$= Tp + Tq$$

Homogeneity: If $\lambda \in \mathbb{F}$ and $p \in \mathcal{P}(\mathbb{R})$, then

$$T(\lambda p) = \int_0^1 (\lambda p)(x)\,dx = \int_0^1 \lambda p(x)\,dx$$
$$= \lambda \int_0^1 p(x)\,dx$$
$$= \lambda Tp.$$

• Multiplication by x^2:

Define $T: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by

$(Tp)(x) = x^2 p(x)$. for all $x \in \mathbb{R}$.

Then $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ because:

Additivity: If $p, q \in \mathcal{P}(\mathbb{R})$, then $\boxed{\text{for all } x \in \mathbb{R},}$

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$$\begin{aligned}(T(p+q))(x) &= x^2(p+q)(x)\\ &= x^2(p(x)+q(x))\\ &= x^2p(x)+x^2q(x)\\ &= (Tp)(x)+(Tq)(x)\\ &= ((Tp)+(Tq))(x)\end{aligned}$$

So $T(p+q) = Tp + Tq$.

Homogeneity: If $\lambda \in \mathbb{F}$ and $p \in \mathcal{P}(\mathbb{R})$, then for all $x \in \mathbb{R}$, we have

$$\begin{aligned}(T(\lambda p))(x) &= x^2(\lambda p)(x)\\ &= x^2(\lambda p(x))\\ &= \lambda x^2 p(x)\\ &= \lambda(Tp)(x).\end{aligned}$$

- From $\mathbb{R}^3$ to $\mathbb{R}^2$: Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x,y,z) = (2x-y+3z,\ 7x+5y-6z).$$

Then $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ because:

Additivity: If $(x,y,z), (\tilde{x},\tilde{y},\tilde{z}) \in \mathbb{R}^3$, then

$$\begin{aligned}T((x,y,z)+(\tilde{x},\tilde{y},\tilde{z})) &= T(x+\tilde{x}, y+\tilde{y}, z+\tilde{z})\\ &= (2(x+\tilde{x})-(y+\tilde{y})+3(z+\tilde{z}),\ 7(x+\tilde{x})+5(y+\tilde{y}) - 6(z+\tilde{z}))\\ &= ((2x-y+3z)+(2\tilde{x}-\tilde{y}+3\tilde{z}),\ (7x+5y-6z)+(7\tilde{x}+5\tilde{y}-6\tilde{z}))\\ &= (2x-y+3z,\ 7x+5y-6z)+(2\tilde{x}-\tilde{y}+3\tilde{z},\ 7\tilde{x}+5\tilde{y}-6\tilde{z})\\ &= T(x,y,z)+T(\tilde{x},\tilde{y},\tilde{z})\end{aligned}$$

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Homogeneity: If $\lambda \in \mathbb{F}$ and $(x,y,z) \in \mathbb{R}^3$, then

$$\begin{aligned} T(\lambda(x,y,z)) &= T(\lambda x, \lambda y, \lambda z) \\ &= (2(\lambda x)-(\lambda y)+3(\lambda z), 7(\lambda x)+5(\lambda y)-6(\lambda z)) \\ &= (\lambda(2x-y+3z), \lambda(7x+5y-6z)) \\ &= \lambda(2x-y+3z, 7x+5y-6z) \\ &= \lambda T(x,y,z). \end{aligned}$$

3.5 Linear maps and basis of domain

Suppose $v_1,\ldots,v_n$ is a basis of V and $w_1,\ldots,w_n \in W$. Then there exists a unique linear map $T: V \to W$ Such that $Tv_j = w_j$ for each $j=1,\ldots,n$.

Proof: First, we will prove that the linear map T exists.

Define $T: V \to W$ by

$$T(c_1v_1+\ldots+c_nv_n) = c_1w_1+\ldots+c_nw_n.$$

for some $c_1,\ldots,c_n \in \mathbb{F}$. Since $v_1,\ldots,v_n$ is a basis of V, every vector in V is uniquely of the form $c_1v_1+\ldots+c_nv_n$. So this map T as we defined above indeed defines a function $T: V \to W$.

Furthermore, for each $j=1,\ldots,n$, if

$$c_j = \begin{cases} 1 & \text{if } j=i \\ 0 & \text{if } j \neq i \text{ (otherwise)}, \end{cases}$$

then T satisfies $Tv_j = w_j$.

Next, we will prove that T is linear, that is $T \in \mathcal{L}(V, W)$.

- Additivity: If $u, v \in V$, then since $v_1, \ldots, v_n$ is a basis of V, we can write $u = a_1 v_1 + \ldots + a_n v_n$

and $v = c_1 v_1 + \ldots + c_n v_n$

for some $a_1, \ldots, a_n, c_1, \ldots, c_n \in \mathbb{F}$. So we have

$$\begin{aligned}
T(u+v) &= T((a_1 v_1 + \ldots + a_n v_n) + (c_1 v_1 + \ldots + c_n v_n)) \\
&= T((a_1 + c_1) v_1 + \ldots + (a_n + c_n) v_n) \\
&= (a_1 + c_1) w_1 + \ldots + (a_n + c_n) w_n \\
&= (a_1 w_1 + c_1 w_1) + \ldots + (a_n w_n + c_n w_n) \\
&= (a_1 w_1 + \ldots + a_n w_n) + (c_1 w_1 + \ldots + c_n w_n) \\
&= T(a_1 v_1 + \ldots + a_n v_n) + T(c_1 v_1 + \ldots + c_n v_n) \\
&= Tu + Tv
\end{aligned}$$

- Homogeneity: If $\lambda \in \mathbb{F}$ and $v \in V$, then we can write

$$v = c_1 v_1 + \ldots + c_n v_n$$

for some $c_1, \ldots, c_n \in \mathbb{F}$. So we have

$$\begin{aligned}
T(\lambda v) &= T(\lambda (c_1 v_1 + \ldots + c_n v_n)) \\
&= T((\lambda c_1) v_1 + \ldots + (\lambda c_n) v_n) \\
&= (\lambda c_1) w_1 + \ldots + (\lambda c_n) w_n \\
&= \lambda c_1 w_1 + \ldots + \lambda c_n w_n \\
&= \lambda (c_1 w_1 + \ldots + c_n w_n) \\
&= \lambda T(c_1 v_1 + \ldots + c_n v_n) \\
&= \lambda T v
\end{aligned}$$

Therefore, $T: V \to W$ satisfies additivity and homogeneity. Therefore, $T \in \mathcal{L}(V, W)$.

Finally, we need to prove that T is unique.
Suppose we have $T \in \mathcal{L}(V, W)$ and T satisfies $Tv_j = w_j$ for each $j = 1, \ldots, n$.

Let $c_1, \ldots, c_n \in \mathbb{F}$ then ~~...~~ the additivity and homogeneity of T gives us:

$$
\begin{aligned}
T(c_1v_1 + \cdots + c_nv_n) &\underset{\text{additivity}}{=} T(c_1v_1) + \cdots + T(c_nv_n) \\
&\underset{\text{homogeneity}}{=} c_1Tv_1 + \cdots + c_nTv_n \\
&= c_1w_1 + \cdots + c_nw_n
\end{aligned}
$$

So T is uniquely determined on $\operatorname{span}(v_1, \ldots, v_n)$.
But $v_1, \ldots, v_n$ is a basis of V, meaning we have
$$\operatorname{span}(v_1, \ldots, v_n) = V$$
So T is uniquely determined on V. $\blacksquare$

Addition and scalar multiplication on $\mathcal{L}(V, W)$

3.6 Definition:

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$.

- The <u>sum</u> $S + T$ is a linear map defined by $(S+T)(v) = Sv + Tv$.
- The <u>product</u> λT is a linear map defined by $(\lambda T)(v) = \lambda(Tv)$.

3.7 $\mathcal{L}(V, W)$ is a vector space

The set $\mathcal{L}(V, W)$ is a vector space with respect to the operations defined in Definition 3.6 of Axler.

Proof: Let ~~[illegible]~~ $R, S, T \in \mathcal{L}(V, W)$ and $a, b \in \mathbb{F}$ be arbitrary.

Commutativity

For all $v \in V$, we have

$$(S+T)v = Sv + Tv$$
$$= Tv + Sv$$
$$= (T+S)v$$

So $S+T = T+S$.

Associativity

For all $v \in V$, we have

$$((R+S)+T)v = (R+S)v + Tv$$
$$= Rv + Sv + Tv$$
$$= Rv + (S+T)v$$
$$= (R+(S+T))v$$

So $(R+S)+T = R+(S+T)$

Additive Identity

Let $0 \in \mathcal{L}(V, W)$ be the zero function. For all $v \in V$, we have

$$(T+0)v = Tv + 0v$$
$$= Tv + 0$$
$$= Tv$$

So $T+0 = T$.

Additive Inverse

Note that we have $-T \in \mathcal{L}(V, W)$. For all $v \in V$, we have

$$(T+(-T))v = Tv + (-T)v$$
$$= Tv - Tv$$
$$= 0$$
$$= 0v.$$

So $T+(-T) = 0$.

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Multiplicative identity

For all $v \in V$, we have

$$(1T)v = 1Tv$$
$$= Tv$$

So $1T = T$

Distributive properties

For all $a \in \mathbb{F}$, for all $v \in V$, we have

$$a(S+T)v = a(Sv + Tv)$$
$$= aSv + aTv$$
$$= (aS + aT)v.$$

So $a(S+T) = aS + aT$.

For all $a, b \in \mathbb{F}$, for all $v \in V$, we have

$$((a+b)T)v = (a+b)Tv$$
$$= aTv + bTv$$
$$= (aT + bT)v$$

So $(a+b)T = aT + bT$

Therefore, $\mathcal{L}(V, W)$ is a vector space with respect to the defined operations.

3.8 Definition:

If ~~S,T ∈ L(V,W)~~, then the product $ST \in \mathcal{L}(U,W)$

$S \in \mathcal{L}(V,W)$, $T \in \mathcal{L}(U,V)$

is defined by

$$(ST)u = S(Tu)$$

for all $u \in U$.

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & V \arrow[r, "S"] & W \end{tikzcd}$$

3.9 Algebraic properties of products of linear maps

- Associativity

  If R, S, T are linear maps such that the product RST makes sense, then $(RS)T = R(ST)$.

- Identity

  If $T \in \mathcal{L}(V,W)$ and $I: V \to V$ is an identity map, then

$$TI = T = IT$$

- Distributive properties

  If $T, T_1, T_2 \in \mathcal{L}(U,V)$ and $S, S_1, S_2 \in \mathcal{L}(V,W)$, then $(S_1 + S_2)T = S_1T + S_2T$

  and

$$S(T_1 + T_2) = ST_1 + ST_2.$$

3.11 Linear maps take 0 to 0

If $T \in \mathcal{L}(V, W)$, then $T(0) = 0$.

<u>Proof</u>: Since T is linear, we can use additivity to get

$$T(0) = T(0+0)$$
additivity of T $$= T(0) + T(0)$$
$$= 2T(0)$$

Therefore $T(0) = 0$, as desired $\boxtimes$