7/8/19 **3A The Vector Space of Linear Maps**

Defn 3.2 A linear map from V to W is a function $T: V \to W$ that satisfies:

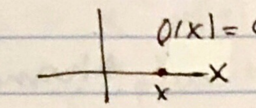
- additivity. $T(u+v) = Tu + Tv$ for all $u, v \in V$
- homogeneity $T(\lambda v) = \lambda Tv$ for all $\lambda \in \mathbb{F}$ and for all $v \in V$

The set of all linear maps $T: V \to W$ is denoted $\mathcal{L}(V,W)$

Eg 3.4 Let $0 \in \mathcal{L}(V,W)$ be the zero map.

defined by $0v = 0$

$0(V)$



$0 \in \mathcal{L}(V,W)$

that is, 0 is a linear map because:

→ additivity: If $u, v \in V$, then
$$0(u+v) = 0 = 0 + 0 = 0u + 0v$$

→ homogeneity: If $u \in U$ and $\lambda \in \mathbb{F}$, then
$$0(\lambda u) = 0 = \lambda \cdot 0 = \lambda 0(u).$$

- identity map: Define $I: V \to V$ by $Iv = v$.

Then $I \in \mathcal{L}(V,V)$ because:

→ additivity: If $u, v \in V$, then
$$I(u+v) = u + v = Iu + Iv$$

→ homogeneity: If $\lambda \in \mathbb{F}$ and $v \in V$, then
$$I(\lambda v) = \lambda v = \lambda Iv.$$

• differentation:

define $D : P(\mathbb{R}) \to P(\mathbb{R})$ by

$$Dp = p'$$

Then $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ because:

→ additivity: If $p, q \in P(\mathbb{R})$, then

$$\begin{aligned} D(p+q) &= (p+q)' \\ &= p' + q' \\ &= Dp + Dq \end{aligned}$$

→ homogeneity: If $\lambda \in \mathbb{F}$ and $p \in D(\mathbb{R})$,

then $$\begin{aligned} D(\lambda p) &= (\lambda p)' \\ &= \lambda p' \\ &= \lambda Dp. \end{aligned}$$

• Integration: Define $T : P(\mathbb{R}) \to \mathbb{R}$ by

$$Tp = \int_0^1 p(x)\,dx$$

then we have $T \in \mathcal{L}(P(\mathbb{R}), \mathbb{R})$ because:

→ additivity: If $p, q \in P(\mathbb{R})$, then

$$\begin{aligned} T(p+q) &= \int_0^1 (p+q)(x)\,dx \\ &= \int_0^1 p(x) + q(x)\,dx \\ &= \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx \\ &= Tp + Tq \end{aligned}$$

→ homogeneity: If $\lambda \in \mathbb{F}$ and $p \in P(\mathbb{R})$, then

$$\begin{aligned} T(\lambda p) = \int_0^1 (\lambda p)(x)\,dx &= \int_0 \lambda p(x)\,dx \\ &= \lambda \int_0^1 p(x)\,dx \\ &= \lambda Tp. \end{aligned}$$

• Multiplication by x^2

Define $T: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by

$$(Tp)(x) = x^2 p(x).$$

for all $x \in \mathbb{R}$. Then $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ because:

→ additivity: If $p, q \in \mathcal{P}(\mathbb{R})$, then $\boxed{\text{for all } x \in \mathbb{R}}$, we have;

$$\begin{aligned}(T(p+q))(x) &= x^2 (p+q)(x) \\ &= x^2 (p(x) + q(x)) \\ &= x^2 p(x) + x^2 q(x) \\ &= (Tp)(x) + (Tq)(x) \\ &= (Tp + Tq)(x).\end{aligned}$$

So $T(p+q) = Tp + Tq$

→ homogeneity: If $\lambda \in \mathbb{F}$ and $p \in \mathcal{P}(\mathbb{R})$, then, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}(T(\lambda p))(x) &= x^2 (\lambda p)(x) \\ &= x^2 (\lambda p(x)) \\ &= \lambda x^2 p(x) \\ &= \lambda (Tp)(x)\end{aligned}$$

· From $\mathbb{R}^3 \to \mathbb{R}^2$: Define $T: \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - y + 3z, \cancel{7x} + 5y - 6z)$$

Then $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ because:

→ additivity: If $(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3$, Then

$$\begin{aligned}T((x, y, z) + (\tilde{x}, \tilde{y}, \tilde{z})) &= T(x + \tilde{x}, y + \tilde{y}, z + \tilde{z}) \\ &= (2(x+\tilde{x}) - (y+\tilde{y}) + 3(z + \tilde{z}), 7(x+\tilde{x}) + 5(y+\tilde{y}) - 6(z+\tilde{z})) \\ &= ((2x - y + 3z) + (2\tilde{x} - \tilde{y} + 3\tilde{z}), (7x + 5y - 6z) + (7\tilde{x} + 5\tilde{y} - 6\tilde{z})) \\ &= (2x - y + 3z, 7x + 5y - 6z) + (2\tilde{x} - \tilde{y} + 3\tilde{z}, 7\tilde{x} + 5\tilde{y} + 6\tilde{z}) \\ &= T(x, y, z) + T(\tilde{x}, \tilde{y}, \tilde{z})\end{aligned}$$

→ homogeneity: If $\lambda \in \mathbb{F}$ and $(x, y, z) + \mathbb{R}^3$, then

$$\begin{aligned}T(\lambda(x, y, z)) &= T(\lambda x, \lambda y, \lambda z) \\ &= (2(\lambda x) - \lambda y) + 3(\lambda z), 7(\lambda x) + 5(\lambda y) - 6(\lambda z)) \\ &= (\lambda(2x - y + 3z), \lambda(7x + 5y - 6z))\end{aligned}$$

→

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Defn 3.5 Linear maps and basis of domain

Suppose $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_n \in W$
then there exists a unique linear map
$T: V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \cdots n$

proof ① First, we will prove that the linear map T exists.
Define $T: V \to W$ by
$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots c_n w_n$$
for some $c_1, \cdots, c_n \in \mathbb{F}$. Since $v_1, \ldots, v_n$ is a basis of V every vector in V is uniquely of the form $c_1 v_1 + \cdots + c_n v_n$. So this map T as we defined above indeed define a function $T: V \to W$.
Furthermore, for each $j = 1, \cdots, n$, if
$$c_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \text{ (otherwise)}, \end{cases}$$
Then T satisfies $Tv_j = w_j$

② Next, we will prove that T is linear; that is, $T \in \mathcal{L}(V, W)$
→ additivity: If $u, v \in V$, then since $v_1, \cdots, v_n$ is a basis of V, we can write
$$u = a_1 v_1 + \cdots + a_n v_n$$
and $v = c_1 v_1 + \cdots + c_n v_n$
for some scalars $a_1, \cdots a_n, c_1, \cdots, c_n \in \mathbb{F}$.
So we have; $T(u+v) = T((a_1 v_1 + \cdots + a_n v_n) + (c_1 v_1 + \cdots + c_n v_n))$
$$= T((a_1 + c_1) v_1 + \cdots + (a_n + c_n) v_n)$$
$$= (a_1 + c_1) w_1 + \cdots + (a_n + c_n) w_n$$

$= (a_1 w_1 + c_1 w_1) + \cdots + (a_n w_n + c_n w_n)$
$= (a_1 w_1 + \cdots + a_n w_n) + (c_1 w_1 + \cdots + c_n w_n)$
$= T(a_1 v_1 + \cdots + a_n v_n) + T(c_1 v_1 + \cdots + c_n v_n)$
$= Tu + Tv.$

→ homogeneity: If $\lambda \in \mathbb{F}$ and $v \in V$, then we can write $v = c_1 v_1 + \cdots + c_n v_n$
for some $c_1, \ldots, c_n \in \mathbb{F}$. So we have

$$
\begin{aligned}
T(\lambda v) &= T(\lambda(c_1 v_1 + \cdots + c_n v_n)) \\
&= T((\lambda c_1) v_1 + \cdots + (\lambda c_n) v_n) \\
&= (\lambda c_1) w_1 + \cdots + (\lambda c_n) w_n \\
&= \lambda c_1 w_1 + \cdots + \lambda c_n w_n \\
&= \lambda (c_1 w_1 + \cdots + c_n w_n) \\
&= \lambda T(c_1 v_1 + \cdots + c_n v_n) \\
&= \lambda Tv.
\end{aligned}
$$

Therefore, $T: V \to W$ satisfies additivity and homogeneity. Therefore, $T \in \mathcal{L}(V, W)$.

Finally(3), we need to prove that T is unique.
Suppose we have $T \in \mathcal{L}(V, W)$ and T satisfies $Tv_j = Tw_j$ for each $j = 1, \ldots, n$.
Let $c_1, \ldots c_n \in \mathbb{F}$, then from homogeneity and additivity of T we have ~~$T(c_j v_j) = c_j T v_j = c_j w_j$~~

~~for each $j = 1, \ldots, n$. And the additivity of~~ T gives us ~~$T(c_1 v_1 + \cdots + c_n v_n) =$~~

gives us: $T(c_1 v_1 + \cdots + c_n v_n) \overset{\text{additivity}}{=} T(c_1 v_1) + \cdots + T(c_n v_n)$
$\overset{\text{homogeneity}}{=} c_1 T v_1 + \cdots c_n T v_n$
$= c_1 w_1 + \cdots + c_n w_n$

So T is uniquely determined on $\operatorname{span}(v_1, \ldots, v_n)$.

But $v_1, \dots, v_n$ is a basis of V, meaning we have

$$\operatorname{span}(v_1, \dots, v_n) = U$$

So T is uniquely determined on V. $\boxed{\square}$

Defn 3.6 **Addition and Scalar Multiplication on $\mathcal{L}(V, W)$**

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$.

- The sum $S+T$ is a linear map defined by
$$(S+T)(v) = Sv + Tv$$
- The product λT is a linear map defined by
$$(\lambda T)(v) = \lambda(Tv)$$

Defn 3.7 **$\mathcal{L}(V, W)$ is a vector space**

The set $\mathcal{L}(V, W)$ is a vector space with respect to the operations defined in Defn 3.6 of Axler

proof: Let $u, v, w \in V$ and $\lambda \in \mathbb{F}$ be arbitrary

→ commutitivity

For all $v \in V$, we have
$$\begin{aligned}(S+T)v &= Sv + Tv \\ &= Tv + Sv \\ &= (T+S)v.\end{aligned}$$
so $S+T = T+S$

→ Associativity

For all $v \in V$, we have
$$\begin{aligned}((R+S)+T)v &= (R+S)v + (T)v \\ &= Rv + Sv + Tv \\ &= Rv + (S+T)v \\ &= (R + (S+T))v\end{aligned}$$
so $(R+S)+T = R+(S+T)$

→ additive identity

Let $0 \in \mathcal{L}(V,W)$ be the zero function.
For all $v \in V$, we have

$$
\begin{aligned}
(T+0)v &= Tv + 0v \\
&= Tv + 0 \\
&= Tv
\end{aligned}
$$

So $T+0 = T$.

→ additive Inverse

Note that we have $-T \in \mathcal{L}(V,W)$. For all $v \in V$, we have

$$
\begin{aligned}
(T + (-T))v &= Tv + (-T)v \\
&= Tv - Tv \\
&= 0 \\
&= 0v
\end{aligned}
$$

So $T + (-T) = 0$

→ Multiplication identity

For all $v \in V$, we have

$$
\begin{aligned}
(1T)v &= 1Tv \\
&= Tv
\end{aligned}
$$

So $1T = T$.

→ Distributive properties

For all $a \in \mathbb{F}$, for all $v \in V$, we have

$$
\begin{aligned}
(a(S+T))v &= a(Sv + Tv) \\
&= aSv + aTv \\
&= (aS + aT)v
\end{aligned}
$$

So $a(S+T) = aS + aT$.

For all $a, b \in \mathbb{F}$, for all $v \in V$, we have

$$
\begin{aligned}
((a+b)T)v &= (a+b)Tv \\
&= aTv + bTv \\
&= (aT + bT)v
\end{aligned}
$$

So $(a+b)T = aT + bT$.

Therefore, $\mathcal{L}(V,W)$ is a vector space with respect to the defined operations

Defn 3.8 If $S \in \mathcal{L}(V,W)$, $T \in \mathcal{L}(U,V)$ then the product $ST \in \mathcal{L}(V,W)$ is defined by

$$(ST)u = S(Tu)$$

for all $u \in U$.

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & V \arrow[r, "S"] & W \end{tikzcd}$$

Thm 3.9 Algebraic properties of linear maps

- associativity
  If R, S, T are linear maps such that the product RST makes sense, then
  $$(RS)T = R(ST)$$
- Identity
  If $T \in \mathcal{L}(V,W)$ and $I : V \to V$ is an identity map, then
  $$TI = T = IT$$
- Distributive properties
  If $T, T_1, T_2 \in \mathcal{L}(U,V)$ and $S_0, S_1, S_2 \in \mathcal{L}(V,W)$,
  Then $(S_1 + S_2)T = S_1T + S_2T$
  and $S(T_1 + T_2) = ST_1 + ST_2$

Thm 3.11 Linear maps take 0 to 0

If $T \in \mathcal{L}(V,W)$ then $T(0) = 0$

proof: Since T is linear, we can use additivety to get

$$T(0) = T(0+0)$$

additivity of $= T(0) + T(0)$

$= 2T(0)$

Therefore $T(0) = 0$, as defined ☐