Since we introduced $u_1, \dots, u_m$ to be a basis of $U_1 \cap U_2$, we can write

$c_1 w_1 + \dots + c_k w_k = d_1 u_1 + \dots + d_m u_m$

for some $d_1, \dots, d_m \in \mathbb{F}$. This means

$c_1 w_1 + \dots + c_k w_k - d_1 u_1 - \dots - d_m u_m = 0$

Since $u_1, \dots, u_m, w_1, \dots, w_k$ is lin indep, all the scalars are zero; $c_1 = 0, \dots, c_k = 0, d_1 = 0, \dots, d_m = 0$. In particular $c_1 = 0, \dots, c_k = 0$.

So the original eqtn,

$a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_j v_j + c_1 w_1 + \dots + c_k w_k = 0$

reduces to

$a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_j v_j = 0$

Since $u_1, \dots, u_m, v_1, \dots, v_j$ is a basis of U_1, it is lin indep, so $a_1 = 0, \dots, a_m = 0, b_1 = 0, \dots, b_j = 0$.

So all scalars are zero. So $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is lin indep. Then by 2.39 of Axler, it is also a basis of $U_1 + U_2$.

Therefore, we have

$$\begin{aligned}\dim(U_1 + U_2) &= m + j + k \\ &= (m+j) + (m+k) - m \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)\end{aligned}$$

7/8/19 week 3 Mon.

3.A The vector space of Linear Maps

3.2 definition: A linear map from V to W is a function $T: V \to W$ that satisfies:

- additivity

  $T(u+v) = Tu + Tv$ for all $u, v \in V$

- homogeneity

  $T(\lambda v) = \lambda T v$ for all $\lambda \in \mathbb{F}$ & for all $v \in V$

The set of all linear maps $T: V \to W$ is denoted $\mathcal{L}(V, W)$.

3.4 Example: Let $0 \in \mathcal{L}(V,W)$ be the zero map defined by $0v = 0$.
$0(v)$

$0 \in \mathcal{L}(V, W)$ that is 0 is a linear map b/c:

additivity: If $u, v \in V$, then
$$\begin{aligned}0(u+v) &= 0 \\ &= 0 + 0 \\ &= 0u + 0v\end{aligned}$$

homogeneity: If $u \in U$ & $\lambda \in \mathbb{F}$ then
$$\begin{aligned}0(\lambda u) &= 0 \\ &= \lambda \cdot 0 \\ &= \lambda\, 0(u)\end{aligned}$$

- identity maps

  Define $I: V \to V$ by

  $Iv = v$

  Then $I \in \mathcal{L}(V, V)$ b/c:

additivity: If $u, v \in V$, then
$$I(u+v) = u+v$$
$$= Iu + Iv$$
homogeneity: If $\lambda \in F$ & $v \in V$, then
$$I(\lambda v) = \lambda v$$
$$= \lambda I v$$

- differentiation

Define $D: P(\mathbb{R}) \to P(\mathbb{R})$ by
$$Dp = p'$$
Then $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ because:

additivity: If $p, q \in P(\mathbb{R})$, then
$$D(p+q) = (p+q)'$$
$$= p' + q'$$
$$= Dp + Dq$$
homogeneity: If $\lambda \in F$ & $p \in P(\mathbb{R})$ then
$$D(\lambda p) = (\lambda p)'$$
$$= \lambda p'$$
$$= \lambda Dp$$

- Integration:

Define $T: P(\mathbb{R}) \to \mathbb{R}$ by
$$Tp = \int_0^1 p(x)\,dx$$
Then we have $T \in \mathcal{L}(P(\mathbb{R}), \mathbb{R})$ b/c:

additivity: If $p, q \in P(\mathbb{R})$, then
$$T(p+q) = \int_0^1 (p+q)(x)\,dx$$
$$= \int_0^1 p(x) + q(x)\,dx$$
$$= \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx$$
$$= Tp + Tq$$
Homogeneity: If $\lambda \in F$ & $p \in P(\mathbb{R})$, then $T(\lambda p) = \int_0^1 (\lambda p)(x)\,dx = \int_0^1 \lambda p(x)\,dx = \lambda \int_0^1 p(x)\,dx = \lambda T p$.

Multiplication by x^2

Define $T: P(\mathbb{R}) \to P(\mathbb{R})$ by $(Tp)(x) = x^2 p(x)$ for all $x \in \mathbb{R}$. Then $T \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ b/c:

Additivity: If $p, q \in P(\mathbb{R})$, then for all $x \in \mathbb{R}$, we have
$$(T(p+q))(x) = x^2(p+q)(x)$$
$$= x^2(p(x) + q(x))$$
$$= x^2 p(x) + x^2 q(x)$$
$$= (Tp)(x) + (Tq)(x)$$
$$= ((Tp) + (Tq))(x)$$
So $T(p+q) = Tp + Tq$

homogeneity: If $\lambda \in \mathbb{F}$ & $p \in \mathcal{P}(\mathbb{R})$, then for $x \in \mathbb{R}$ we have

$$\begin{aligned}(T(\lambda p))(x) &= x^2(\lambda p)(x) \\ &= x^2(\lambda p(x)) \\ &= \lambda x^2 p(x) \\ &= \lambda (Tp)(x)\end{aligned}$$

- From $\mathbb{R}^3$ to $\mathbb{R}^2$: Define $T: \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z)$$

Then $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ b/c:

Additivity: If $(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3$, then

$$\begin{aligned}T((x, y, z) + (\tilde{x}, \tilde{y}, \tilde{z})) &= T(x+\tilde{x}, y+\tilde{y}, z+\tilde{z}) \\ &= (2(x+\tilde{x}) - (y+\tilde{y}) + 3(z+\tilde{z}), 7(x+\tilde{x}) + 5(y+\tilde{y}) - 6(z+\tilde{z})) \\ &= ((2x - y + 3z) + (2\tilde{x} - \tilde{y} + 3\tilde{z}), 7x + 5y - 6z) - (7\tilde{x} + 5\tilde{y} - 6\tilde{z})) \\ &= (2x - y + 3z, 7\end{aligned}$$

Homogeneity: If $\lambda \in \mathbb{F}$ & $(x, y, z) \in \mathbb{R}^3$, then

$$\begin{aligned}T(\lambda(x, y, z)) &= T(\lambda x, \lambda y, \lambda z) \\ &= (2(\lambda x) - (\lambda y) + 3(\lambda z),\end{aligned}$$

3.5 linear Maps & basis of Domain

Suppose $v_1, \ldots, v_n$ is a basis of V & $w_1, \ldots, w_n \in W$

Then there exists a unique linear map $T: V \to W$ such that

$Tv_j = w_j$ for each $j = 1, \ldots, n$

Proof: First, we will prove that the linear map T exists.

Define $T: V \to W$ by

$T(c_1 v_1 + \ldots + c_n v_n) = c_1 w_1 + \ldots + c_n w_n$ for some $c_1, \ldots, c_n \in \mathbb{F}$

Since $v_1, \ldots, v_n$ is a basis of V, every vector in V is uniquely of the form $c_1 v_1 + \ldots + c_n v_n$. So this map T as we defined above indeed define a function $T: V \to W$

Furthermore for each $j = 1, \ldots, n$, if $c_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \text{ (otherwise)} \end{cases}$

then T satisfies $Tv_j = w_j$

Next we will prove that T is linear; that is $T \in \mathcal{L}(V, W)$.

- Additivity: If $u, v \in V$, then since $v_1, \ldots, v_n$ is a basis of V, we can write $u = a_1 v_1 + \ldots + a_n v_n$ & $v = c_1 v_1 + \ldots + c_n v_n$ for some $a_1, \ldots, a_n, c_1, \ldots, c_n \in \mathbb{F}$. So we have $T(u+v) =$

$$\begin{aligned}&T((a_1 v_1 + \ldots + a_n v_n) + (c_1 v_1 + \ldots + c_n v_n)) \\ &= T((a_1 + c_1) v_1 + \ldots + (a_n + c_n) v_n) \\ &= (a_1 + c_1) w_1 + \ldots + (a_n + c_n) w_n \\ &= (a_1 w_1 + c_1 w_1) + \ldots + (a_n w_n + c_n w_n)\end{aligned}$$

$= (a_1 w_1 + \dots + a_n w_n) + (c_1 w_1 + \dots + c_n w_n)$
$= T(a_1 v_1 + \dots + a_n v_n) + T(c_1 v_1 + \dots + c_n v_n)$
$= Tu + Tv$

Homogeneity: If $\lambda \in F$ & $v \in V$, then we can write
$v = c_1 v_1 + \dots + c_n v_n$ for some $c_1, \dots, c_n \in F$. So we have
$$T(\lambda v) = T(\lambda(c_1 v_1 + \dots + c_n v_n))$$
$$= T((\lambda c_1) v_1 + \dots + (\lambda c_n) v_n)$$
$$= (\lambda c_1) w_1 + \dots + (\lambda c_n) w_n$$
$$= \lambda c_1 w_1 + \dots + \lambda c_n w_n$$
$$= \lambda(c_1 w_1 + \dots + c_n w_n)$$
$$= \lambda T(c_1 v_1 + \dots + c_n v_n)$$
$$= \lambda T v$$

Finally we need to prove that T is unique
suppose we have $T \in \mathcal{L}(V, W)$ & T satisfies $Tv_j = w_j$ for each $j = 1, \dots$
let $c_1, \dots, c_n \in F$, then the additivity & homogeneity of T gives us:
$$T(c_1 v_1 + \dots + c_n v_n) \underset{\text{additivity}}{=} T(c_1 v_1) + \dots + T(c_n v_n)$$
$$\underset{\text{homogeneity}}{=} c_1 T v_1 + \dots + c_n T v_n$$
$$= c_1 w_1 + \dots + c_n w_n$$
So T is uniquely determined on $\text{span}(v_1, \dots, v_n)$. But $v_1, \dots, v_n$ is a basis of V, meaning we have $\text{span}(v_1, \dots, v_n)$
so T is uniquely determined on V.

3.6 Definition

Addition & scalar multi on $\mathcal{L}(V, W)$

Suppose $S, T \in \mathcal{L}(V, W)$ & $\lambda \in F$

- The sum $S + T$ is a linear map defined by $(S + T)(v) = Sv + Tv$
- The product λT is a linear map defined by $(\lambda T)(v) = \lambda(Tv)$

3.7 $\mathcal{L}(V, W)$ is a vector space

The set $\mathcal{L}(V, W)$ is a vector space w/ respect to the operations defined in Def. 3.6 of Axler.

Proof: let $R, S, T \in \mathcal{L}(V, W)$ & $a, b \in F$ be arbitrary

commutativity for all $v \in V$, we have
$$(S + T)v = Sv + Tv$$
$$= Tv + Sv$$
$$= (T + S)v$$
So $S + T = T + S$

Associativity

For all $v \in V$, we have

$$\begin{aligned}((R+S)+T)v &= (R+S)v + Tv \\ &= Rv + Sv + Tv \\ &= Rv + (S+T)v \\ &= (R+(S+T))v \quad \text{so } (R+S)+T = R+(S+T).\end{aligned}$$

Additive Identity

Let $0 \in \mathcal{L}(V, W)$ be the zero function. For all $v \in V$, we have

$$\begin{aligned}(T+0)v &= Tv + 0v \\ &= Tv + 0 \\ &= Tv \quad \text{so } T+0 = T\end{aligned}$$

Add. Inverse

Note that we have $-T \in \mathcal{L}(V, W)$. For all $v \in V$, we have

$$\begin{aligned}(T+(-T))v &= Tv + (-T)v \\ &= Tv - Tv \\ &= 0 \\ &= 0v \quad \text{so } T+(-T) = 0\end{aligned}$$

Multiplicative Identity

For all $v \in V$, we have

$$\begin{aligned}(1T)v &= 1Tv \\ &= Tv\end{aligned}$$

so $1T = T$

Distributive Property

For all $a \in \mathbb{F}$, for all $v \in V$, we have

$$\begin{aligned}(a(S+T))v &= a(Sv+Tv) \\ &= aSv + aTv \\ &= (aS + aT)v\end{aligned}$$

so $a(S+T) = aS + aT$

For all $a, b \in \mathbb{F}$, for all $v \in V$, we have

$$\begin{aligned}((a+b)T)v &= (a+b)Tv \\ &= aTv + bTv \\ &= (aT+bT)v\end{aligned}$$

so $(a+b)T = aT + bT$

Therefore, $\mathcal{L}(V, W)$ is a vector space w/ respect to the defined operations.

3.8 Definition

If $S \in \mathcal{L}(V, W)$, $T \in \mathcal{L}(U, V)$, then the product $ST \in \mathcal{L}(U, W)$ is defined by $(ST)u = S(Tu)$ for all $u \in U$

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & V \arrow[r, "S"] & W \end{tikzcd}$$

3.9 Algebraic Prop. of products of linear maps

- Associativity

  If R, S, T are linear maps such that the product RST makes sense, then $(RS)T = R(ST)$

- Identity

  If $T \in \mathcal{L}(V, W)$ & $I : V \to V$ is an Identity map, then $TI = T = IT$

- Distributive

  If $T, T_1, T_2 \in \mathcal{L}(U, V)$ & $S, S_1, S_2 \in \mathcal{L}(V, W)$, then $(S_1 + S_2)T = S_1T + S_2T$ & $S(T_1 + T_2) = ST_1 + ST_2$

3.11 linear maps take 0 to 0

If $T \in \mathcal{L}(V, W)$, then $T(0) = 0$

Proof: since T is linear, we can use additivity to get

$$T(0) = T(0+0) = T(0) + T(0) = 2T(0)$$

Therefore $T(0) = 0$, as defined.

7/9/19 week 3 Tues.

3.B Null Spaces & Ranges

3.12 Definition

If we have $T \in \mathcal{L}(V, W)$, then the null space of T is the subset of V consisting of vectors in V that T maps to 0: $\text{null } T = \{v \in V : Tv = 0\}$

3.13 example

- consider the zero map $0 \in \mathcal{L}(V, W)$. For all $v \in V$, we have $0v = 0$.

  Therefore, $\text{null } 0 = \{v \in V : 0v = 0\} = V$

- Define $P \in \mathcal{L}(\mathbb{C}^3, \mathbb{C})$ by $P(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$

  Then we have

  $$\text{null } P = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : P(z_1, z_2, z_3) = 0\} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + 2z_2 + 3z_3 = 0\}$$

  The basis of null P is $(-2, 1, 0), (-3, 0, 1)$