Lecture 08
07-08-19

Section 3A : The Vector Space of Linear Maps

3.2 Definition : A linear map from V to W is a function $T: V \to W$ that satisfies ;

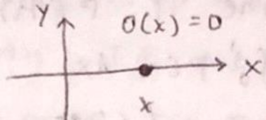
- additivity

  $T(u+v) = Tu + Tv$ for all $u, v \in V$
- homogeneity

  $T(\lambda v) = \lambda T v$ for all $\lambda \in \mathbb{F}$ and for all $v \in V$

The set of all linear maps $T: V \to W$ is denoted $\mathcal{L}(V, W)$

3.4 Examples : Let $0 \in \mathcal{L}(V, W)$ be the zero map defined by $0v = 0$
$0(v)$



$0 \in \mathcal{L}(V, W)$, that is, 0 is a linear map because :

- additivity : If $u, v \in V$, then

$$0(u+v) = 0$$
$$= 0 + 0$$
$$= 0u + 0v$$

- homogeneity : If $u \in U$ and $\lambda \in \mathbb{F}$, then

$$0(\lambda u) = 0$$
$$= \lambda \cdot 0$$
$$= \lambda \cdot 0(u)$$

→ identity maps

Define $I: V \to V$ by

$$Iv = v$$

Then $I \in \mathcal{L}(V, V)$ because :

- additivity : If $u, v \in V$, then

$$I(u+v) = u + v$$
$$= Iu + Iv$$

- homogeneity : If $\lambda \in \mathbb{F}$ and $v \in V$, then

$$I(\lambda v) = \lambda v$$
$$= \lambda I v$$

→ differentiation

Define $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by

$$Dp = p'$$

Then $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ because:

- additivity: If $p, q \in \mathcal{P}(\mathbb{R})$, then

$$\begin{aligned} D(p+q) &= (p+q)' \\ &= p' + q' \\ &= Dp + Dq \end{aligned}$$

- homogeneity: If $\lambda \in \mathbb{F}$ and $p \in \mathcal{P}(\mathbb{R})$, then

$$\begin{aligned} D(\lambda p) &= (\lambda p)' \\ &= \lambda p' \\ &= \lambda Dp \end{aligned}$$

→ Integration: Define $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ by

$$Tp = \int_0^1 p(x)\,dx$$

Then we have $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ because

- additivity: If $p, q \in \mathcal{P}(\mathbb{R})$ then

$$\begin{aligned} T(p+q) &= \int_0^1 (p+q)(x)\,dx \\ &= \int_0^1 p(x) + q(x)\,dx \\ &= \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx \\ &= Tp + Tq \end{aligned}$$

- homogeneity: If $\lambda \in \mathbb{F}$ and $p \in \mathcal{P}(\mathbb{R})$, then

$$\begin{aligned} T(\lambda p) &= \int_0^1 (\lambda p)(x)\,dx \\ &= \int_0^1 \lambda p(x)\,dx \\ &= \lambda \int_0^1 p(x)\,dx \\ &= \lambda Tp \end{aligned}$$

→ Multiplication by x^2

Define $T: P(\mathbb{R}) \to P(\mathbb{R})$ by

$$(Tp)(x) = x^2 p(x)$$

for all $x \in \mathbb{R}$. Then $T \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ because:

- additivity: If $p, q \in P(\mathbb{R})$, then $\boxed{\text{for all } x \in \mathbb{R}}$, we have

$$\begin{aligned}(T(p+q))(x) &= x^2(p+q)(x) \\ &= x^2(p(x) + q(x)) \\ &= x^2 p(x) + x^2 q(x) \\ &= (Tp)(x) + (Tq)(x) \\ &= ((Tp) + (Tq))(x)\end{aligned}$$

so $T(p+q) = Tp + Tq$

- homogeneity: If $\lambda \in \mathbb{F}$ and $p \in P(\mathbb{R})$, then, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}(T(\lambda p))(x) &= x^2(\lambda p)(x) \\ &= x^2(\lambda p(x)) \\ &= \lambda x^2 p(x) \\ &= \lambda (Tp)(x)\end{aligned}$$

From $\mathbb{R}^3$ to $\mathbb{R}^2$: Define $T: \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z)$$

Then $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ because:

- Additivity: If $(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3$, then

$$\begin{aligned}T((x,y,z) + (\tilde{x}, \tilde{y}, \tilde{z})) &= T(x+\tilde{x}, y+\tilde{y}, z+\tilde{z}) \\ &= (2(x+\tilde{x}) - (y+\tilde{y}) + 3(z+\tilde{z}), 7(x+\tilde{x}) + 5(y+\tilde{y}) - 6(z+\tilde{z})) \\ &= ((2x-y+3z) + (2\tilde{x}-\tilde{y}+3\tilde{z}), (7x+5y-6z) + (7\tilde{x}+5\tilde{y}-6\tilde{z})) \\ &= (2x-y+3z, 7x+5y-6z) + (2\tilde{x}-\tilde{y}+3\tilde{z}, 7\tilde{x}+5\tilde{y}+6\tilde{z}) \\ &= T(x,y,z) + T(\tilde{x}, \tilde{y}, z)\end{aligned}$$

- Homogeneity: If $\lambda \in \mathbb{F}$ and $(x, y, z) \in \mathbb{R}^3$, then

$$\begin{aligned}T(\lambda(x,y,z)) &= T(\lambda x, \lambda y, \lambda z) \\ &= (2(\lambda x) - (\lambda y) + 3(\lambda z), 7(\lambda x) + 5(\lambda y) - 6(\lambda z)) \\ &= (\lambda(2x - y + 3z), \lambda(7x + 5y + 6z))\end{aligned}$$

$= \lambda((2x - y + 3z), (7x + 5y + 6z))$

$= \lambda T(x, y, z)$

3.5 Linear maps and basis of domain

Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n \in W$. Then there exists a unique linear map $T: V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \dots, n$

Proof: First we will prove that the linear map T exists. Define $T: V \to W$ by

$$T(c_1 v_1 + \dots + c_n v_n) = c_1 w_1 + \dots + c_n w_n$$

for som $c_1, \dots, c_n \in \mathbb{F}$. Since $v_1, \dots, v_n$ is a basis of V, every vector in V is uniquely the form $c_1 v_1 + \dots + c_n v_n$. So this map T as we defined above indeed define a function $T: V \to W$

Futhermore, for each $j = 1, \dots, n$, if

$$c_j = \begin{cases} 1 & \text{if } j = v \\ 0 & \text{if } j \neq v \text{ (otherwise)} \end{cases},$$

then T satisfies $Tv_j = w_j$.

Next, we will prove that T is linear, that is $T \in \mathcal{L}(V, W)$

- Additivity: If $u, v \in V$, then since $v_1, \dots, v_n$ is a basis of V, we can write

$$u = a_1 v_1 + \dots + a_n v_n$$

and

$$v = c_1 v_1 + \dots + c_n v_n$$

for some $a_1, \dots, a_n, c_1, \dots, c_n \in \mathbb{F}$ so we have

$$\begin{aligned} T(u+v) &= T((a_1 v_1 + \dots + a_n v_n) + (c_1 v_1 + \dots + c_n v_n)) \\ &= T((a_1 + c_1) v_1 + \dots + (a_n + c_n) v_n) \\ &= (a_1 + c_1) w_1 + \dots + (a_n + c_n) w_n \\ &= (a_1 w_1 + c_1 w_1) + \dots + (a_n w_n + c_n w_n) \end{aligned}$$

$$= T(a_1 v_1 + \cdots + c_n v_n) + T(c_1 v_1 + \cdots + c_n v_n)$$
$$= Tu + Tv$$

• homogeneity: If $\lambda \in \mathbb{F}$ and $v \in V$, then we can write

$$v = c_1 v_1 + \cdots + c_n v_n$$

for some $c_1, \ldots, c_n \in \mathbb{F}$ So we have

$$\begin{aligned}
T(\lambda v) &= T(\lambda(c_1 v_1 + \cdots c_n v_n)) \\
&= T((\lambda c_1) v_1 + \cdots + (\lambda c_n) v_n) \\
&= (\lambda c_1) w_1 + \cdots + (\lambda c_n) w_n \\
&= \lambda c_1 w_1 + \cdots + \lambda c_n w_n \\
&= \lambda (c_1 w_1 + \cdots + c_n w_n) \\
&= \lambda T(c_1 v_1 + \cdots + c_n v_n) \\
&= \lambda T v
\end{aligned}$$

Therefore, $T : V \to W$ satisfies additivity and homogeneity

Therefore $T \in \mathcal{L}(V, W)$

Finally we need to prove that T is unique

Suppose we have $T \in \mathcal{L}(V, W)$ and T satisfies $Tv_j = w_j$ for each $j = 1, \ldots, n$

then the additivity and homogeneity of T gives us

$$\begin{aligned}
T(c_1 v_1 + \cdots + c_n v_n) &\overset{\text{additivity}}{=} T(c_1 v_1) + \cdots + T(c_n v_n) \\
&\overset{\text{homogeneity}}{=} c_1 T v_1 + \cdots + c_n T v_n \\
&= c_1 w_1 + \cdots + c_n w_n
\end{aligned}$$

Addition and scalar multiplication on $\mathcal{L}(V, W)$

3.6 Definition

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$

- The sum $S + T$ is a linear map defined by $(S+T)(v) = Sv + Tv$
- The product λT is a linear map defined by $(\lambda T)(v) = \lambda (Tv)$

3.7 $\mathcal{L}(V, W)$ is a vector space

The set $\mathcal{L}(V, W)$ is a vector space with respect to the operations defined in Definition 3.6 of Axler

Proof: Let $R, S, T \in \mathcal{L}(V, W)$ and $a, b \in \mathbb{F}$ be arbitrary

- commutivity

  For all $v \in V$ we have

  $$\begin{aligned}(S+T)v &= Sv + Tv \\ &= Tv + Sv \\ &= (T+S)v\end{aligned}$$

  so $S+T = T+S$

- associativity

  for all $v \in V$ we have

  $$\begin{aligned}((R+S)+T)v &= (R+S)v + Tv \\ &= Rv + Sv + Tv \\ &= Rv + (S+T)v \\ &= (R+(S+T))v\end{aligned}$$

  so $(R+S)+T = R+(S+T)$

- additive identity

  Let $0 \in \mathcal{L}(V, W)$ be the zero function. For all $v \in V$ we have

  $$\begin{aligned}(T+0)v &= Tv + 0v \\ &= Tv + 0 \\ &= Tv\end{aligned}$$

  so $T+0 = T$

- additive inverse

  Note that we have $-T \in \mathcal{L}(V, W)$. For all $v \in V$, we have

  $$\begin{aligned}(T+(-T))v &= Tv + (-T)v \\ &= Tv - Tv \\ &= 0 \\ &= 0v\end{aligned}$$

  So $T+(-T) = 0$

- multiplicative identity

  For all $v \in V$ we have

  $$\begin{aligned}(1T)v &= 1 \cdot Tv \\ &= Tv\end{aligned}$$

  So $1T = T$

• distributive properties

For all $a \in \mathbb{F}$ for all $v \in V$, we have

$$\begin{aligned}(a(S+T))v &= a(Sv+Tv)\\ &= aSv + aTv \\ &= (aS+aT)v\end{aligned}$$

so $a(S+T) = aS + aT$

For all $a, b \in \mathbb{F}$, for all $v \in V$, we have

$$\begin{aligned}((a+b)T)v &= (a+b)Tv \\ &= aTv + bTv \\ &= (aT+bT)v\end{aligned}$$

so $(a+b)T = aT + bT$

Therefore $\mathcal{L}(V, W)$ is a vector space with respect to the defined operations

3.8 Definition

If $S \in \mathcal{L}(V,W)$, $T \in \mathcal{L}(U,V)$, then the product $ST \in \mathcal{L}(U,W)$ is defined by

$$(ST)u = S(Tu)$$

for all $u \in U$

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & V \arrow[r, "S"] & W \end{tikzcd}$$

3.9 Algebraic properties of products of linear maps

• associativity

If R, S, T are linear maps such that the product RST makes sense, then

$$(RS)T = R(ST)$$

• identity

If $T \in \mathcal{L}(V, W)$ and $I: V \to V$ is an identity map, then

$$TI = T = IT$$

• distributive properties

If $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$, then

$$(S_1 + S_2)T = S_1T + S_2T$$

and

$$S(T_1+T_2) = ST_1 + ST_2$$

3.11 Linear maps take 0 to 0

If $T \in \mathcal{L}(V, W)$, then $T(0) = 0$

Proof:

Since T is linear, we can use additivity to get

$$\begin{aligned} T(0) &= T(0 + 0) \\ &\overset{\text{additivity of } T}{=} T(0) + T(0) \\ &= 2T(0) \end{aligned}$$

Therefore, $T(0) = 0$, as desired