3.A The Vector Space of Linear Maps

3.2 Definition

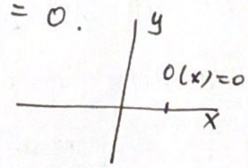
A linear map from V to W is a function $T: V \to W$ that satisfies:

- additivity: $T(u+v) = Tu + Tv$ for all $u, v \in V$
- homogeneity: $T(\lambda v) = \lambda Tv$ for all $\lambda \in \mathbb{F}$ and for all $v \in V$.

The set of all linear maps $T: V \to W$ is denoted $\mathcal{L}(V, W)$.

3.4 Examples :

(1) Zero map: Let $0 \in \mathcal{L}(V, W)$ be the zero map defined by $0v = 0$.
$0(v)$



~~ete~~ $0 \in \mathcal{L}(V, W)$

this is, 0 is a linear map because:

- additivity: If $u, v \in V$, then
$$0(u+v) = 0 = 0+0 = 0u + 0v$$
- homogeneity: If $u \in U$ and $\lambda \in \mathbb{F}$, then ~~then~~
$$0(\lambda u) = 0 = \lambda \cdot 0 = \lambda 0(u).$$

(2) Identity map: Define $I: V \to V$ by $Iv = v$.

Then $I \in \mathcal{L}(V,V)$ because:

- additivity: If $u, v \in V$, then
$$I(u+v) = u+v = Iu + Iv$$
- homogeneity: If $\lambda \in \mathbb{F}$ and $v \in V$, then
$$I(\lambda v) = \lambda v = \lambda Iv.$$

(3) Differentiation.

Define $D: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $Dp = p'$.

Then $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ because

- additivity: If $p, q \in \mathcal{P}(\mathbb{R})$, then
$$D(p+q) = (p+q)' = p' + q' = Dp + Dq$$
- homogeneity: If $\lambda \in \mathbb{F}$ and $p \in \mathcal{P}(\mathbb{R})$, then
$$D(\lambda p) = (\lambda p)' = \lambda p' = \lambda Dp$$

(4) Integration.

Define $T: \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ by $Tp = \int_0^1 p(x)\,dx$.

Then we have $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ because:

- Additivity: If $p, q \in \mathcal{P}(\mathbb{R})$, then
$$T(p+q) = \int_0^1 (p+q)(x)\,dx = \int_0^1 p(x) + q(x)\,dx = \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx = Tp + Tq.$$
- Homogeneity: If $\lambda \in \mathbb{F}$ and $p \in \mathcal{P}(\mathbb{R})$, then
$$T(\lambda p) = \int_0^1 (\lambda p)(x)\,dx = \int_0^1 \lambda p(x)\,dx = \lambda \int_0^1 p(x)\,dx = \lambda Tp.$$

(5) Multiplication by x^2.

Define $T: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x)$

For all $x \in \mathbb{R}$. Then $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ because:

- Additivity: If $p, q \in \mathcal{P}(\mathbb{R})$, then $\boxed{\text{For all } x \in \mathbb{R}}$, we have
$$(T(p+q))(x) = x^2(p+q)(x) = x^2(p(x) + q(x)) = x^2 p(x) + x^2 q(x) = (Tp)(x) + (Tq)(x) = ((Tp) + (Tq))(x)$$
So $T(p+q) = Tp + Tq$.
- Homogeneity: If $\lambda \in \mathbb{F}$ and $p \in \mathcal{P}(\mathbb{R})$, then, for all $x \in \mathbb{R}$, we have
$$(T(\lambda p))(x) = x^2(\lambda p)(x) = x^2(\lambda p(x)) = \lambda x^2 p(x) = \lambda (Tp)(x)$$

(6) <u>From $\mathbb{R}^3$ to $\mathbb{R}^2$</u>:

Define $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x,y,z) = (2x-y+3z,\ 7x+5y-6z)$

Then $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ because:

- Additivity: If $(x,y,z), (\tilde{x},\tilde{y},\tilde{z}) \in \mathbb{R}^3$, then

$$
\begin{aligned}
&T((x,y,z)+(\tilde{x},\tilde{y},\tilde{z})) \cancel{\in \mathbb{R}^3} \\
&= T(x+\tilde{x},\ y+\tilde{y},\ z+\tilde{z}) \\
&= (2(x+\tilde{x})-(y+\tilde{y})+3(z+\tilde{z}),\ 7(x+\tilde{x})+5(y+\tilde{y})-6(z+\tilde{z})) \\
&= ((2x-y+3z)+(2\tilde{x}-\tilde{y}+3\tilde{z}),\ (7x+5y+6z)+(7\tilde{x}+5\tilde{y}-6\tilde{z})) \\
&= (2x-y+3z, 7x+5y-6z)+(2\tilde{x}-\tilde{y}+3\tilde{z}, 7\tilde{x}+5\tilde{y}-6\tilde{z}) \\
&= T(x,y,z)+T(\tilde{x},\tilde{y},\tilde{z}).
\end{aligned}
$$

- Homogeneity: If $\lambda \in \mathbb{F}$ and $(x,y,z) \in \mathbb{R}^3$, then

$$
\begin{aligned}
T(\lambda(x,y,z)) &= T(\lambda x, \lambda y, \lambda z) \\
&= (2(\lambda x)-(\lambda y)+3(\lambda z),\ 7(\lambda x)+5(\lambda y)-6(\lambda z)) \\
&= (\lambda(2x-y+3z),\ \lambda(7x+5y-6z)) \\
&= \lambda(2x-y+3z,\ 7x+5y-6z) \\
&= \lambda T(x,y,z).
\end{aligned}
$$

3.5 Linear maps and basis of domain

Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n \in W$

Then there exists a unique linear map $T: V \to W$ such that

$$Tv_j = w_j \quad \text{for each } j = 1, \dots, n.$$

<u>Proof</u>: First, we will prove that the linear map T exists.

Define $T: V \to W$ by $T(c_1v_1 + \dots + c_nv_n) = c_1w_1 + \dots + c_nw_n$ for some $c_1, \dots, c_n \in \mathbb{F}$. Since $v_1, \dots, v_n$ is a basis of V, every vector in V is uniquely of the form $c_1v_1 + \dots + c_nv_n$.

So this map T as we defined above indeed define a function $T: V \to W$.

Furthermore, for each $j = 1, \dots, n$,

if $c_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$ (otherwise),

then T satisfies $Tv_j = w_j$.

Next, we will prove that T is linear, this is $T \in \mathcal{L}(V, W)$.

- Additivity: If $u, v \in V$, then since $v_1, \dots, v_n$ is a basis of V, we can write
$$u = a_1 v_1 + \cdots + a_n v_n$$
and
$$v = c_1 v_1 + \cdots + c_n v_n$$
for some $a_1, \dots, a_n, c_1, \dots, c_n \in \mathbb{F}$. So we have
$$\begin{aligned} T(u+v) &= T((a_1 v_1 + \cdots + a_n v_n) + (c_1 v_1 + \cdots + c_n v_n)) \\ &= T((a_1 + c_1) v_1 + \cdots + (a_n + c_n) v_n) \\ &= (a_1 + c_1) w_1 + \cdots + (a_n + c_n) w_n \\ &= (a_1 w_1 + c_1 w_1) + \cdots + (a_n w_n + c_n w_n) \\ &= (a_1 w_1 + \cdots + a_n w_n) + (c_1 w_1 + \cdots + c_n w_n) \\ &= T(a_1 v_1 + \cdots + a_n v_n) + T(c_1 v_1 + \cdots + c_n v_n) \\ &= Tu + Tv. \end{aligned}$$

- Homogeneity: If $\lambda \in \mathbb{F}$ and $v \in V$, then we can write
$$v = c_1 v_1 + \cdots + c_n v_n$$
for some $c_1, \dots, c_n \in \mathbb{F}$. So we have
$$\begin{aligned} T(\lambda v) &= T(\lambda(c_1 v_1 + \cdots + c_n v_n)) \\ &= T((\lambda c_1) v_1 + \cdots + (\lambda c_n) v_n) \\ &= (\lambda c_1) w_1 + \cdots + (\lambda c_n) w_n \\ &= \lambda c_1 w_1 + \cdots + \lambda c_n w_n \\ &= \lambda (c_1 w_1 + \cdots + c_n w_n) \\ &= \lambda T(c_1 v_1 + \cdots + c_n v_n) \\ &= \lambda Tv. \end{aligned}$$

Therefore, $T: V \to W$ satisfies additivity and homogeneity.
Therefore, $T \in \mathcal{L}(V, W)$.

Finally, we need to prove that T is unique.

Suppose we have $T \in \mathcal{L}(V, W)$ and T satisfies $Tv_j = w_j$ for each $j = 1, \dots, n$. $\forall c_1, \dots, c_n \in \mathbb{F}$, then the additivity and homogeneity of T gives us:

$$\begin{aligned} T(c_1 v_1 + \cdots + c_n v_n) &\overset{\text{additivity}}{=} T(c_1 v_1) + \cdots + T(c_n v_n) \\ &\overset{\text{homogeneity}}{=} c_1 T v_1 + \cdots + c_n T v_n \\ &= c_1 w_1 + \cdots + c_n w_n \end{aligned}$$

So T is uniquely determined on $\operatorname{span}(v_1, \dots, v_n)$.
But $v_1, \dots, v_n$ is a basis of V, meaning we have $\operatorname{span}(v_1, \dots, v_n) = V$.
So T is uniquely determined on V.

Addition and Scalar multiplication on $\mathcal{L}(V, W)$

3.6 Definition

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$

- The sum $S+T$ is a linear map defined by $(S+T)(v) = Sv + Tv$
- The product λT is a linear map defined by $(\lambda T)(v) = \lambda(Tv)$

3.7 $\mathcal{L}(V, W)$ is a vector space

The set $\mathcal{L}(V, W)$ is a vector space with respect to the operators defined in Definition 3.6 of Axler.

Proof: Let $R, S, T \in \mathcal{L}(V, W)$ and $a, b \in \mathbb{F}$ be arbitrary.

- Commutivity For all $v \in V$, we have

$$\begin{aligned} (S+T)v &= Sv + Tv \\ &= Tv + Sv \\ &= (T+S)v \end{aligned}$$

So $S+T = T+S$

- Associativity For all $v \in V$, we have

$$
\begin{aligned}
((R+S)+T)v &= (R+S)v + Tv \\
&= Rv + Sv + Tv \\
&= Rv + (S+T)v \\
&= (R+(S+T))v
\end{aligned}
$$

So $(R+S)+T = R+(S+T)$

- Additive identity Let $0 \in \mathcal{L}(V,W)$ be the zero vector.

For all $v \in V$, we have

$$
\begin{aligned}
(T+0)v &= Tv + 0v \\
&= Tv + 0 \\
&= Tv.
\end{aligned}
$$

So $T+0=T$

- Additive inverse Note that we have $-T \in \mathcal{L}(V,W)$.

For all $v \in V$, we have

$$
\begin{aligned}
(T+(-T))v &= Tv + (-T)v \\
&= Tv - Tv \\
&= 0 \\
&= 0v.
\end{aligned}
$$

So $T+(-T)=0$.

- Multiplicative identity For all $v \in V$, we have

$$
\begin{aligned}
(1T)v &= 1Tv \\
&= Tv
\end{aligned}
$$

So $1T = T$.

- Distributive properties For all $a \in \mathbb{F}$, for all $v \in V$, we have

$$
\begin{aligned}
(a(S+T))v &= a(Sv + Tv) \\
&= aSv + aTv \\
&= (aS+aT)v
\end{aligned}
$$

So $a(S+T) = aS + aT$.

For all $a, b \in \mathbb{F}$, for all $v \in V$, we have

$$
\begin{aligned}
((a+b)T)v &= (a+b)Tv \\
&= aTv + bTv \\
&= (aT+bT)v
\end{aligned}
$$

So $(a+b)T = aT+bT$.

Therefore, $\mathcal{L}(V,W)$ is a vector space with respect to the defined operations.

3.8 Definition

If $S \in \mathcal{L}(V,W)$, $T \in \mathcal{L}(V,W)$, then the <u>product</u> $ST \in \mathcal{L}(U,W)$ is defined by $(ST)u = S(Tu)$ for all $u \in U$

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & V \arrow[r, "S"] & W \end{tikzcd}$$

3.9 Algebras (properties of products of linear maps)

- <u>Associativity</u> If R, S, T are linear maps such that the product RST makes sence, then $(RS)T = R(ST)$.
- <u>Identity</u> If $T \in \mathcal{L}(V,W)$ and $I: V \to V$ is an identity map, then $TI = T = IT$
- <u>Distributive properties</u> If $T, T_1, T_2 \in \mathcal{L}(U,V)$ and $S, S_1, S_2 \in \mathcal{L}(V,W)$, then $(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$.

3.11 Linear maps take 0 to 0

If $T \in \mathcal{L}(V,W)$, then $T(0) = 0$.

<u>Proof</u>: Since T is linear, we can use additivity to get

$$\begin{aligned} T(0) &= T(0+0) \\ &\overset{\text{additivity of } T}{=} T(0) + T(0) \\ &= 2T(0) \end{aligned}$$

Therefore, $T(0) = 0$, as defines.