3.B) Null Spaces & Ranges

3.12 $\boxed{\text{Def}}$

If we have $T \in \mathcal{L}(V,W)$, then the $\boxed{\text{nullspace}}$ of T is the subset of V consisting of vectors in V that T maps to 0:

$$\text{null } T = \{v \in V : Tv = 0\}$$

3.13 Examples

Consider the zero map $0 \in \mathcal{L}(V,W)$.

$\forall v \in V,\ 0v = 0$

Therefore, null $0 = \{v \in V : 0v = 0\} = V$

- Define $\varphi \in \mathcal{L}(\mathbb{C}^3, \mathbb{C})$ by

$$\varphi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$$

then we have

$$\text{null } \varphi = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \varphi(z_1, z_2, z_3) = 0\}$$
$$= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + 2z_2 + 3z_3 = 0\}$$

The basis of null φ is $(-2,1,0), (-3,0,1)$

$$(z_1, z_2, z_3) = (-2z_2 - 3z_3, z_2, z_3)$$
$$= z_2(-2,1,0) + z_3(-3,0,1)$$

So $(-2,1,0), (-3,0,1)$ span null φ

Then prove $(-2,1,0), (-3,0,1)$ is l.I

So $(-2,1,0), (-3,0,1)$ is a basis of null φ.

• Let $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ be the differentiation map defined by $Dp = p'$

Then,

$$\text{null } D = \{p \in P(\mathbb{R}) : p'(z) = 0 \ \forall z \in \mathbb{F}\}$$
$$= \{p \in P(\mathbb{R}) : p'(z) = c \ \forall z \in \mathbb{F}, \text{ for some constant } c\}$$
$$= \{p \in P(\mathbb{R}) : p \text{ is a constant function}\}$$

• Define $T \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ by

$$(Tp)(x) = x^2 p(x)$$

$\forall x \in \mathbb{R}$

Then we have,

$$\text{null } T = \{p \in P(\mathbb{R}) : Tp = 0\}$$
$$= \{p \in P(\mathbb{R}) : (Tp)(x) = 0 \quad \forall x \in \mathbb{R}\}$$
$$= \{p \in P(\mathbb{R}) : x^2 p(x) = 0 \ \forall x \in \mathbb{R}\}$$
$$= \{p \in P(\mathbb{R}) : p(x) = 0 \ \forall x \in \mathbb{R}\}$$
$$= \{p \in P(\mathbb{R}) : p = 0\} = \{0\}$$

3.14 Null space is a subspace

Suppose we have $T \in \mathcal{L}(V, W)$, then null space of T is a subspace of V.

Add Identit Proof: Since we have $T \in \mathcal{L}(V, W)$, it follows $T : V \to W$ is a linear map. By 3.11, we have $T(0) = 0$.

Therefore, $0 \in \text{null } T$

Closed under add:

Suppose we have $u, v \in \text{null } T$, then $Tu = 0$ & $Tv = 0$

$$T(u+v) = Tu + Tv$$
$$= 0 + 0$$
$$= 0$$

So, $u + v \in \text{null } T$.

Closed under scalar multil.

Suppose we have $u \in \text{null } T$ & $\lambda \in \mathbb{F}$. Then $Tu = 0$

So we have, $T(\lambda u) = \lambda Tu = \lambda \, 0 = 0$ so $\lambda u \in \text{null } T$.

This means null T is a subspace of V

3.15 $\boxed{\text{Def}}$)
A function $T: V \to W$ is called injective if $Tu = Tv$ implies $u = v$, $\forall u, v \in V$

3.16 Injectivity is equivalent to null space equals $\{0\}$.
Let $T \in \mathcal{L}(V, W)$ Then T is injective IFF null $T = \{0\}$.

Proof: Forward direction: If T is injective, then null $T = \{0\}$

Part 1
Suppose T is injective. Since T is also linear, we have $T(0) = 0$ which means $0 \in$ null T, and so $\{0\} \subset$ null T.
We will prove $T \subset \{0\}$
Suppose $v \in$ null T,
Then $Tv = 0$. But we also have $T(0) = 0$
Therefore, $Tv = T(0)$, $(Tv = 0 = T(0))$
Since T is injective, $v = 0$
In other words, $v \in \{0\}$.
So null $T \subset \{0\}$ So we get the set equality null $T = \{0\}$

Backward direction: If null $T = \{0\}$, then T is injective.

2
Suppose null $T = \{0\}$. Suppose $u, v \in V$ satisfy $Tu = Tv$
Then we have $0 = Tu - Tv = T(u - v)$
Therefore, $u - v \in$ null T but null $T = \{0\}$
so $u - v \in \{0\}$, which implies $u - v = 0$, so $u = v$. Therefore T is injective ∎

Range & surjectivity

3.17 $\boxed{\text{Def}}$
The range of a function $T: V \to W$ is a subset of W consisting of all vectors of the form Tv for some $v \in V$

and is denoted:

$$\text{range } T = \{Tv : v \in V\}$$

3.18 Example

Consider the zero map $0: V \to W$. Then $0v = 0 \ \forall v \in V$.
So we have, $\text{range } 0 = \{0v : v \in V\}$
$= \{0 : v \in V\}$
$= \{0\}$

- Define $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ by $T(x,y) = (2x, 5y, x+y)$

Then we have $\text{range } T = \{T(x,y) : (x,y) \in \mathbb{R}^2\}$
$= \{(2x, 5y, x+y) : (x, y) \in \mathbb{R}^2\}$

A basis of range T is $(2,0,1), (0,5,1)$.
$T(x,y) = (2x, 5y, x+y) = (2x, 0, x) + (0, 5y, y)$
$= x(2,0,1) + y(0,5,1)$

So $(2,0,1), (0,5,1)$ spans range T.
Show that $(2,0,1), (0,5,1)$ is $\mathcal{LI}$.
Then $(2,0,1), (0,5,1)$ is a basis of range T.

- Let $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ be the differentiation map defined by $Dp = p'$

for each polynomial $q \in \mathcal{P}(\mathbb{R})$, there exists a polynomial $p \in \mathcal{P}(\mathbb{R})$ that satisfies $p' = q$
So we have $\text{range } D = \{Dp : p \in \mathcal{P}(\mathbb{R})\}$
$= \{p' : p \in \mathcal{P}(\mathbb{R})\}$
$= \{q : q \in \mathcal{P}(\mathbb{R})\}$
$= \mathcal{P}(\mathbb{R})$.

3.19 The range is a subspace

Proof: If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W.

Add. Identi

Suppose we have $T \in \mathcal{L}(V, W)$. Then by 3.11, we have $T(0) = 0$. So $0 \in$ range T.

Closed under add.

Suppose $w_1, w_2 \in$ range T. Then $w_1 = Tv_1$ & $w_2 = Tv_2$ for some $v_1, v_2 \in V$

So we have $T(v_1+v_2) = Tv_1 + Tv_2 = w_1 + w_2$.
Since $v_1+v_2 \in V$, it follows that we have $w_1+w_2 \in$ range T. ✓

Closed under multi

Suppose $\lambda \in \mathbb{F}$ & $w \in$ range T, this $w = Tv$ for some $v \in V$.
So we have $T(\lambda v) = \lambda Tv = \lambda w$
since $\lambda v \in V$ it follows that we have $\lambda w \in$ range T
So range T is a subspace of W.

☆ 3.22 Fundamental Theorem of Linear Maps

Suppose V is a finite-dim vector space & $T \in \mathcal{L}(V, W)$. Then range T is finite-dim. and dim V = dim null T + dim range T.

3.20 def)

A function $T: V \to W$ is surjective if its range equals W; that is, range $T = W$

Proof: Let $u_1, \dots, u_m$ be a basis of null T. Then dim(null T) = m. Also, $u_1, \dots, u_m$ is a l.I. list. By 2.33, we can extend the list to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V. Then dim $V = m+n$.

So we need to just show that range T is finite-dim, with dim(range T) $= n$.
To accomplish this goal, we need to show that $Tv_1, \dots, Tv_n$ is a basis of range T.
Let $v \in V$ be arbitrary. Since $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis of V, it spans V.
So we can write:
$v = a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_nv_n$
for some $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$. So we have
$Tv = T(a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_nv_n)$

(Additivity of T) $= T(a_1 u_1) + \cdots + T(a_m u_m) + T(b_1 v_1) + \cdots + T(b_n v_n)$

(Homogeneity of T) $= a_1 Tu_1 + \cdots + a_m Tu_m + b_1 Tv_1 + \cdots + b_n Tv_n$

($u_1, \ldots, u_m \in \text{null } T$ $u_1, \ldots, u_m$ is a basis of null T) $=$

$a_1 0 + \cdots + a_m 0 + b_1 Tv_1 + \cdots + b_n Tv_n = b_1 Tv_1 + \cdots + b_n Tv_n$

There, the list $Tv_1, \ldots, Tv_n$ spans range T.

Since we found a list that spans range T, we conclude that range T is finite-dim.

- Now show that $Tv_1, \ldots, Tv_n$ is ℓI

Suppose $c_1, \ldots, c_n \in \mathbb{F}$ that satisfy

$$c_1 Tv_1 + \cdots + c_n Tv_n = 0$$

Then we have $0 = c_1 Tv_1 + \cdots + c_n Tv_n$

$= T(c_1 v_1) + \cdots + T(c_n v_n)$

Therefore, $c_1 v_1 + \cdots + c_n v_n \in \text{null } T$.

Since $u_1, \ldots, u_m$ is a basis of null T, it spans null T, so we can write $c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots + d_m u_m$ for some $d_1, \ldots, d_m \in \mathbb{F}$ so we get

$$-d_1 u_1 + \cdots + d_m u_m + c_1 v_1 + \cdots + c_n v_n = 0$$

But $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V, so it is ℓI in V. So all scalars are zero:

$-d_1 = 0, \ldots -d_m = 0, c_1 = 0, \ldots, c_n = 0.$

In particular,

$$c_1 = 0, \ldots, c_n = 0$$

So $Tv_1, \ldots, Tv_n$ is ℓI.

So it is a basis of range T.

So $\dim(\text{range } T) = n$.

- Summarily,

$\dim V = m + n$

$\dim(\text{null } T) = m$

$\dim(\text{range } T) = n$

Therefore, $\dim V = m + n = \dim(\text{null } T) + \dim(\text{range } T)$

3.23. Amap smaller dim. space is not injective
Suppose V & W are finite-dim. vector spaces. that satisfi $\dim V > \dim W$. The no linear map from V to W is injective.

Proof: Suppose $T \in \mathcal{L}(V, W)$.
By Fund. thm. of linear maps. (3.22),

$$\begin{aligned} \dim(\operatorname{null} T) &= \dim V - \dim(\operatorname{range} T) \\ &\geq \dim V - \dim W \\ &> \dim W - \dim W \\ &= 0. \end{aligned}$$

range of T is a subspace of W by 2.38. ($\dim(\operatorname{range} T) \leq \dim W$)

So $\operatorname{null} T \neq \{0\}$ By 3.16, T is not injective ▨

3.24 A map to larger dim. space is not surjective

Suppose V & W are finite-dim vector spaces that satisfy $\dim V < \dim W$. Then no linear map from V to W is surjective.

Proof: Let $T \in \mathcal{L}(V, W)$ then, by Fund. thm. of linear maps. (3.22)

$$\begin{aligned} \dim(\operatorname{range} T) &= \dim V - \dim \operatorname{null} T \\ &\leq \dim - 0 \\ &= \dim V \\ &< \dim W. \end{aligned}$$

By Exercise 2.C1 of axler, $\operatorname{range} T \neq W$.
Therefore, T is not surjective ▨

3.25 Example

Fixed positive integers m, n.

Consider the homogeneous system of linear equations

A system of m equations

$$\begin{cases} \sum_{k=1}^{n} A_{1,k} x_k = 0 \\ \sum_{k=1}^{n} A_{m,k} x_k = 0 \end{cases}$$

j-th equation

$$\sum_{k=1}^{n} A_{j,k} x_k = 0$$

for some $A_{j,k} \in \mathbb{F}$, for $j = 1, \dots, m$ & for $k = 1, \dots, n$

Rephrase in terms of a linear map the question of whether this system of equations has a nonzero solution.

Proof: Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$\underbrace{T(x_1, \dots, x_n)}_{n \text{ coordinates}} = \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{m,k} x_k\right)}_{m \text{ coordinates}}$$

Then the equation

$$T(\underbrace{0, \dots, 0}_{n}) = (\underbrace{0, \dots, 0}_{m})$$

is equivalent to the homogeneous system of equations ∎

Note that $(x_1, \dots, x_n) = (0, \dots, 0)$ is a solution to the homogeneous system of equations.

This is equivalent to saying,

$$T(\underbrace{0, \dots, 0}_{n}) = (\underbrace{0, \dots, 0}_{m})$$

There exist nonzero solutions $x_1, \dots, x_n$ to the homogeneous system of equations IFF null $T \neq \{0\}$

In other words IFF $\exists\, x_1, \dots, x_n$ not ALL zero such that $(x_1, \dots, x_n) \in$ null T

3.26 Homogeneous sys of linear equations

A homogeneous sys. of linear equations with more var. than equations has nonzero solutions.

Example:
$\sum_{k=1}^{n} A_{1,k} x_k = 0$
$\sum_{k=1}^{n} A_{2,k} x_k = 0$
and x_1, x_2, x_3 solve the above sys. of equations then at least one is nonzero.

Proof: Again, define $T: \mathbb{F}^n \to \mathbb{F}^m$ by
$T(x_1, \ldots, x_n) = \sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k)$

Then we have homo. sys. of m linear eqs with var. $x_1, \ldots, x_n$. Since there are more vars then there are eqs, we have:
$\dim \mathbb{F}^n = n > m = \dim \mathbb{F}^m$
By 3.23, $T: \mathbb{F}^n \to \mathbb{F}^m$ is not injective.
By 3.16, null $T \neq \{0, \ldots, 0\}$. So there exists nonzero.

3.27 Example

Rephrase in terms of a linear map the question of whether the inhomogeneous sys of linear eqs.

$$\sum_{k=1}^{n} A_{1,k} x_k = c_1$$

$$\sum_{k=1}^{n} A_{m,k} x_k = c_m,$$

for any $A_{j,k} \in \mathbb{F}$. ($j = 1, \ldots, m$ & $k = 1, \ldots, n$) and for some $c_1, \ldots, c_m \in \mathbb{F}$, has no solution.

Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$\underbrace{T(x_1, \ldots, x_m)}_{n-\text{coord}} \quad \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k\right)}_{m-\text{coord.}}$$

Then the equation $T(x_1, \ldots, x_n) = (c_1, \ldots, c_m)$ is equivalent to the inhomogeneous sys. of eqs. ∎

3.29 Inhomo. sys. of linear eqs.

An inhomo. sys. of linear eqs. with more equations than the variables has no solution for some choice of constant terms.

Proof: Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$\underbrace{T(x_1, \dots, x_n)}_{n\text{-coord}} = \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{m,k} x_k\right)}_{m\text{-coord.}}$$

Since there are more eqs. than vars,

$$\dim \mathbb{F}^n = n < m = \dim \mathbb{F}^m$$

By 3.24, $T: \mathbb{F}^n \to \mathbb{F}^m$ is NOT surjective.

So range $T \neq \mathbb{F}^m$

So the exists $(c_1, \dots, c_m \in \mathbb{F}^m \setminus \text{range } T)$ such that,

$$T(x_1, \dots, x_n) \neq (c_1, \dots, c_m)$$

So we have

$$\sum_{k=1}^{n} A_{1,k} x_k \neq c_1$$

$$\sum_{k=1}^{n} A_{m,k} x_k \neq c_n$$

which means the sys. of inhomogeneous eqs. with this $c_1, \dots, c_m$ does NOT contain any solutions $\boxtimes$