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3.B Null space and ranges.

3.12 Definition.

If we have $T \in \mathcal{L}(V,W)$, then the null space of T is the subset of V consisting of vectors in V that T maps to 0.

$\text{null}\, T = \{v \in V : Tv = 0\}$

3.12 Examples

- Consider the zero map $0 \in \mathcal{L}(V,W)$. For all $v \in V$, we have $0v = 0$.

  Therefore, $\text{null}\, 0 = \{v \in V : 0v = 0\}$
  $= V$

- Define $\varphi \in \mathcal{L}(\mathbb{C}^3, \mathbb{C})$ by $\varphi = (z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$

  Then we have

  $$\text{null}\, \varphi = \{(z_1, z_2, z_3) \in \mathbb{C}^3 ; \varphi(z_1, z_2, z_3) = 0\}$$
  $$= \{(z_1, z_2, z_3) \in \mathbb{C}^3, z_1 + 2z_2 + 3z_3 = 0\}$$

  The basis of null φ is $(-2,1,0), (-3,0,1)$.

  $(z_1, z_2, z_3) = (-2z_2 - 3z_3, z_2, z_3)$
  $= z_2(-2,1,0) + z_3(-3,0,1)$

  $\therefore (-2,1,0), (-3,0,1)$ spans null φ.

  Then prove that $(-2,1,0), (-3,0,1)$ is linearly indepent.

  So $(-2,1,0), (-3,0,1)$ is a basis of null φ

- Let $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ be the differentiation map defined by $Dp = p'$

Then we have

$$\begin{aligned} \text{null } D &= \{p \in P(\mathbb{R}) : p'(z) = 0 \text{ for all } z \in \mathbb{F}\} \\ &= \{p \in P(\mathbb{R}) : p'(z) = c \text{ for all } z \in \mathbb{F}. \text{ for some constant } c\} \\ &= \{p \in P(\mathbb{R}) : p \text{ is a constant function}\} \end{aligned}$$

- Define $T \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ by

$$(Tp)(x) = x^2 p(x) \quad \text{for all } x \in \mathbb{R}$$

Then we have

$$\begin{aligned} \text{null } T &= \{p \in P(\mathbb{R}) : Tp = 0\} \\ &= \{p \in P(\mathbb{R}) : (Tp)(x) = 0 \text{ for all } x \in \mathbb{R}\} \\ &= \{p \in P(\mathbb{R}) : x^2 p(x) = 0 \text{ for all } x \in \mathbb{R}\} \\ &= \{p \in P(\mathbb{R}) : p(x) = 0 \text{ for all } x \in \mathbb{R}\} \\ &= \{p \in P(\mathbb{R}) : p = 0) \\ &= \{0\} \end{aligned}$$

The only polynomial satisfies is the 0.

3.14 Null space is a subspace

Suppose we have $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

- additive identity

Proof: Since we have $T \in \mathcal{L}(V, W)$, it follows that $T: V \Rightarrow W$ is a linear map. By 3.11 of Axler, we have $T(0) = 0$. Therefore, $0 \in$ null T.

- Closed under addition

suppose we have $u, v \in$ null T. Then we have $Tu = 0$. $Tv = 0$

So we have: $T(u+v) = Tu + Tv$

$= 0 + 0 = 0$

$\therefore u+v \in \text{null}\, T$

- Closed under scalar multiplication.

Suppose we have $u \in \text{null}\, T$ and $\lambda \in \mathbb{F}$, then $Tu = 0$

So we have $T(\lambda u) = \lambda Tu$

$= \lambda \cdot 0 = 0$

$\therefore \lambda u \in \text{null}\, T$

$\therefore$ We conclude that null T is a subspace of V.

3.15 Definition

A function $T: V \to W$ is called injective

if $Tu = Tv$ implies $u = v$, for all $u, v \in V$

3.16 Injectivity is equivalent to null space equals $\{0\}$.

Let $T \in \mathcal{L}(V, W)$. Then T is injective iff $\text{null}\, T = \{0\}$.

Proof: Forward direction: if T is injective, then $\text{null}\, T = \{0\}$.

Suppose T is injective. Since T is also linear, we have

$T(0) = 0$, which means $0 \in \text{null}\, T$, and so $\{0\} \subset \text{null}\, T$.

We will prove $\text{null}\, T \subset \{0\}$. Suppose $v \in \text{null}\, T$.

Then $Tv = 0$. But we also have $T(0) = 0$.

$\therefore$ $Tv = T(0)$. $(Tv = 0 = T(0))$

Since T is injective, $v = 0$. In other word, $v \in \{0\}$,

So $\text{null}\, T \subset \{0\}$, So we get the set equivalently $\text{null}\, T = \{0\}$.

Backward direction: If $\text{null}\, T = \{0\}$, then T is injective.

Suppose $\operatorname{null} T = \{0\}$. Suppose $u, v \in V$ satisfy $Tu = Tv$.

Then we have $0 = Tu - Tv$

$= T(u-v)$

Therefore, $u - v \in \operatorname{null} T$. But $\operatorname{null} T = \{0\}$. So $u - v \in \{0\}$,

which implies $u - v = 0$, or equivalently, $u = v$, $\therefore T$ is injective.

Range and surgectivity

3.17 Definition

The range of a function $T: V \to W$ is a subset of W consisting of all vectors of the form Tv for some $v \in V$ and is denoted

$$\operatorname{range} T = \{Tv : v \in V\}$$

3.18 Example

Consider the zero map $0: V \to W$. Then $0v = 0$ for all $v \in V$.

So we have
$$\begin{aligned}\operatorname{range} 0 &= \{0v : v \in V\} \\ &= \{0 : v \in V\} \\ &= \{0\}\end{aligned}$$

- Define $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ by $T(x, y) = (2x, 5y, x+y)$

Then we have

$$\begin{aligned}\operatorname{range} T &= \{T(x, y) : (x, y) \in \mathbb{R}^2\} \\ &= \{(2x, 5y, x+y) : (x, y) \in \mathbb{R}^2\}\end{aligned}$$

A basis of range T is $(2, 0, 1)$, $(0, 5, 1)$

$T(x, y) = x(2, 0, 1) + y(0, 5, 1)$

$\therefore$ $(2,0,1), (0,5,1)$ spans range T.

Show that $(2,0,1),(0,5,1)$ is also linearly independent.

Then $(2,0,1)(0,5,1)$ is a basis of range T.

- Let $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ be the differentiation map defined by $Dp = p'$

For each polynomial $q \in P(\mathbb{R})$, there exists a polynomial $p \in P(\mathbb{R})$ that satisfies $p' = q$

So we have

$$\begin{aligned} \text{range}\, D &= \{Dp : p \in P(\mathbb{R})\} \\ &= \{p' : p \in P(\mathbb{R})\} \\ &= \{q : q \in P(\mathbb{R})\} \\ &= P(\mathbb{R}) \end{aligned}$$

3.19 The range is a subspace

Proof: • Additive identity

Suppose we have $T \in \mathcal{L}(V,W)$. Then by 3.11 of Axler, we have $T(0)=0$. So $0 \in$ range T.

• Closed under addition:

Suppose $w_1, w_2 \in$ range T. Then $w_1 = Tv_1$ and $w_2 = Tv_2$ for some $v_1, v_2 \in V$.

So we have
$$\begin{aligned} T(v_1 + v_2) &= Tv_1 + Tv_2 \\ &= w_1 + w_2 \end{aligned}$$

Since $v_1 + v_2 \in V$, it follows that we have $w_1 + w_2 \in$ range T.

• Closed under scalar multiplication

Suppose $\lambda \in \mathbb{F}$ and $w \in$ range T. Then $w = Tv$ for some

$v \in V$.

So we have:

$$T(\lambda v) = \lambda Tv$$
$$= \lambda Tw$$

Since $\lambda v \in V$, it follows that we have $\lambda w \in \text{range}\, T$.

So range T is a subspace of W.

3.22 Fundamental Theorem of linear maps.

Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V,W)$

Then range T is finite-dimensional and

$$\dim V = \dim \text{null}\, T + \dim \text{range}\, T$$

Proof: Let $u_1, \ldots, u_m$ be a basis of null T. Then $\dim(\text{null}\, T) = m$. Also, $u_1, \ldots, u_m$ is a linear independent list. By 2.33, we can extend this list to a basis: $u_1, \ldots, u_m, v_1, \ldots, v_n$ of V.

Then $\dim V = m+n$.

So we need to just show that range T is finite-dimensional with $\dim(\text{range}\, T) = n$. To accomplist this goal, we need to show that $Tv_1, \ldots, Tv_n$ is a basis of range T.

Let $v \in V$ be arbitrary. Since $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V, it spans V.

So we can write:

$$v = a_1 u_1 + \ldots + a_m u_m + b_1 v_1 + \ldots + b_n v_n.$$

for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$. So we have.

$$Tv = T(a_1 v_1 + \ldots + a_m v_m + b_1 v_1 + \ldots + b_n v_n)$$

$$\overset{\text{additivity of } T}{=} T(a_1v_1)+\dots+T(a_mv_m)+T(b_1v_1)+\dots+T(b_nv_n)$$

$$\overset{\text{homogeneity of } T}{=} a_1T(v_1)+\dots+a_mTv_m+b_1Tv_1+\dots+b_nTv_n$$

$$= a_1 0+\dots+a_m 0+b_1Tv_1+\dots+b_nTv_n$$

$$= b_1Tv_1+\dots+b_nTv_n$$

$\therefore$ the list $Tv_1,\dots,Tv_n$ spans range T.

Since we found a list that spans range T, we conclude that range T is finite-dimensional.

Now we will show that $Tv_1,\dots,Tv_n$ is linearly independent

Suppose $c_1,\dots,c_n \in \mathbb{F}$ that satisfy

$$c_1Tv_1+\dots+c_nTv_n=0$$

Then we have

$$0= c_1Tv_1+\dots+c_nTv_n$$
$$=T(c_1v_1)+\dots+T(c_nv_n)$$
$$=T(c_1v_1+\dots+c_nv_n)$$

$\therefore c_1v_1+\dots+c_nv_n \in \operatorname{null} T$

Since $u_1,\dots,u_m$ is a basis of null T, it spans null T,

so we can write $c_1v_1+\dots+c_nv_n=d_1v_1+\dots+d_nv_m$

for some $d_1,\dots,d_m \in \mathbb{F}$. So we get

$-d_1v_1-\dots-d_nv_m+c_1v_1+\dots+c_nv_n=0$

But $u_1,\dots,u_m,v_1,\dots,v_n$ is a basis of V. so it is linearly independent in V.

So all scalars are zero:

$-d_1=0,\dots,-d_m=0,\ c_1=0,\dots,c_n=0$

In particular, $c_1=0,\dots,c_n=0$

So $Tv_1,\dots,Tv_n$ is linearly independent.

So it is a basis of range T.

$\therefore$ $\dim(\operatorname{range} T) = n$

In summary, we have:

$\dim V = m + n$

$\dim(\operatorname{null} T) = m$

$\dim(\operatorname{range} T) = n$

$\therefore$ $\dim V = m + n = \dim(\operatorname{null} T) + \dim(\operatorname{range} T)$

as desired.

3.23 A map to a smaller dimensional space is not injective

Suppose V and W are finite-dimensional vector spaces that satisfy $\dim V > \dim W$. Then no linear map from V to W is injective.

Proof:

Suppose by contradiction that we have $T \in \mathcal{L}(V, W)$

Then, by the Thm of linear maps (3.22) we have

$$\begin{aligned}\dim(\operatorname{null} T) &= \dim V - \dim(\operatorname{range} T)\\ &\geq \dim V - \dim W\\ &> \dim W - \dim W = 0\end{aligned}$$

So $\operatorname{null} T \neq \{0\}$. By 3.16 of Axler, T is not injective.

3.24 A map to a larger space is not surjective.

Suppose V and W are finite-dimensional vector spaces that satisfy $\dim V < \dim W$. Then no linear map from

V to W is surjective.

Proof:

Let $T \in \mathcal{L}(V, W)$. Then by the 3.22, we have

$$\begin{aligned} \dim(\text{range } T) &= \dim V - \dim \text{null } T \\ &\leq \dim V - 0 \\ &= \dim V \end{aligned}$$

By Exercise 2.C1 of Axler, range $T \neq W$.

$\therefore$ T is not surjective.

3.25 Example

Consider the homogeneous system of linear equations

A system of m equations
$$\begin{cases} \sum_{k=1}^{n} A_{1,k} x_k = 0 \\ \vdots \\ \sum_{k=1}^{n} A_{m,k} x_k = 0 \end{cases}$$

for some $A_{j,k} \in \mathbb{F}$, for $j=1,\dots,m$ and for $k=1,\dots,n$.

Rephrase in terms of a linear map the question of whether this system of equations has a nonzero solution.

Proof:

Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$\underbrace{T(x_1,\dots,x_n)}_{n \text{ coordinates}} = \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{m,k} x_k\right)}_{m \text{ coordinates}}$$

Then the equation

$$T(\underbrace{0,\dots,0}_{n}) = (\underbrace{0,\dots,0}_{m})$$

is equivalent to the homogeneous system of equations.

Note that

$(x_1,\dots,x_n) = (0,\dots,0)$ is a solution to the homogeneous system of equations. This is equivalent to saying

$$T(\underbrace{0,\dots,0}_{n}) = (\underbrace{0,\dots,0}_{m})$$

There exist nonzero solutions $x_1,\dots,x_n$ to the homogeneous system of equations iff $\operatorname{null} T \neq \{0\}$.

In other words, iff there exist $x_1,\dots,x_n$, not all zero such that $(x_1, \ldots, x_n) \in \operatorname{null} T$.

3.26 Homogeneous system of linear equations.

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Proof: Again, define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(x_1, \ldots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k\right)$$

Then we have a homogeneous system of m linear equations with n variables $x_1, \ldots, x_n$. Since there are more variables, then there are equations, we have

$$\dim \mathbb{F}^n = n > m = \dim \mathbb{F}^m$$

By 3.23, $T: \mathbb{F}^n \to \mathbb{F}^m$ is not injective

By 3.16, $\operatorname{null} T \neq \{0, \ldots, u\}$ so there exist nonzero solutions of the system of equations.

3.27 Example

Rephrase in terms of a linear map the question of whether the inhomogeneous system of linear equations.

$$\sum_{k=1}^{n} A_{1,k} x_k = C_1$$

$$\vdots$$

$$\sum_{k=1}^{n} A_{m,k} x_k = C_m$$

for any $A_{j,k} \in \mathbb{F}$ ($j = 1, \ldots, m$ and $k = 1, \ldots, m$) and for some $C_1, \ldots, C_m \in \mathbb{F}$ has no solution.

Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$T(x_1, \dots, x_n)$ $\left(\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{m,k} x_k\right)$

n-coordinates m-coordinates

Then the equation: $T(x_1, \dots, x_n) = (c_1, \dots, c_n)$

is equivalent to the inhomogeneous system of equations

3.29 Inhomogeneous system of linear equations.

An inhomogeneous system of linear equations with more equations than the variables has no solution for some choice of constant terms.

Proof: Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(x_1, \dots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{m,k} x_k\right)$$

Since there are more equations than there are variables, we have $\dim \mathbb{F}^n = n < m = \dim \mathbb{F}^m$.

By 3.24, $T: \mathbb{F}^n \to \mathbb{F}^m$ is not surjective. In other words, range $T \neq \mathbb{F}^m$

So there exist $c_1, \dots, c_m \in \mathbb{F}^m \setminus \operatorname{range} T$

s.t. $T(x_1, \dots, x_n) \neq (c_1, \dots, c_m)$

So we have $\sum_{k=1}^{n} A_{1,k} x_k \neq c_1$

$$\vdots$$

$$\sum_{k=1}^{n} A_{m,k} x_k \neq c_m$$

which means the system of inhomogeneous equations with this choice of $c_1, \dots, c_m$ does not contain any solutions.