

7/9 3.3 Null spaces and Ranges

3.12 Definition:

If we have $T \in \mathcal{L}(V, W)$, then the null space of T is the subset of V consisting of vectors in V that T maps to 0 .

$$\text{null } T = \{v \in V : Tv = 0\}$$

3.13 Examples:

• Consider the zero map $0 \in \mathcal{L}(V, W)$. For all $v \in V$, we have

$$0v = 0$$

Therefore, $\text{null } 0 = \{v \in V : 0v = 0\} = V$

• Define $\phi \in \mathcal{L}(\mathbb{C}^3, \mathbb{C})$ by $\phi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$

Then we have

$$\begin{aligned} \text{null } \phi &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \phi(z_1, z_2, z_3) = 0\} \\ &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + 2z_2 + 3z_3 = 0\} \end{aligned}$$

The basis of $\text{null } \phi$ is $(-2, 1, 0), (-3, 0, 1)$.

$$\begin{aligned} (z_1, z_2, z_3) &= (-2z_2 - 3z_3, z_2, z_3) \\ &= (-2z_2, z_2, 0) + (-3z_3, 0, z_3) \\ &= z_2(-2, 1, 0) + z_3(-3, 0, 1) \end{aligned}$$

So $(-2, 1, 0), (-3, 0, 1)$ spans $\text{null } \phi$

Then prove that $(-2, 1, 0), (-3, 0, 1)$ is linearly independent

So $(-2, 1, 0), (-3, 0, 1)$ is a basis of $\text{null } \phi$.

- Let $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ be the differentiation map defined by

$$Dp = p'$$

Then we have

$$\begin{aligned} \text{null } D &= \{p \in P(\mathbb{R}) : p'(z) = 0 \text{ for all } z \in \mathbb{F}\} \\ &= \{p \in P(\mathbb{R}) : p'(z) = c \text{ for all } z \in \mathbb{F}, \text{ for some constant } c\} \\ &= \{p \in P(\mathbb{R}) : p \text{ is a constant function}\} \end{aligned}$$

- Define $T \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ by
- $$(Tp)(x) = x^2 p(x)$$
- for all $x \in \mathbb{R}$.

Then we have

$$\begin{aligned} \text{null } T &= \{p \in P(\mathbb{R}) : Tp = 0\} \\ &= \{p \in P(\mathbb{R}) : (Tp)(x) = 0 \text{ for all } x \in \mathbb{R}\} \\ &= \{p \in P(\mathbb{R}) : x^2 p(x) = 0 \text{ for all } x \in \mathbb{R}\} \end{aligned}$$

The only polynomial that satisfies

$$x^2 p(x) = 0$$

is $p = 0$.

$$\begin{aligned} &\rightarrow \{p \in P(\mathbb{R}) : p(x) = 0 \text{ for all } x \in \mathbb{R}\} \\ &= \{p \in P(\mathbb{R}) : p = 0\} \end{aligned}$$

3.14 Null space is a subspace

Suppose we have $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V .

Proof: Additive Identity

Since we have $T \in \mathcal{L}(V, W)$, it follows that $T: V \rightarrow W$ is a linear map. By 3.11 of Axler, we have $T(0) = 0$. Therefore, $0 \in \text{null } T$

• Closed under Addition

Suppose we have $u, v \in \text{null } T$. Then we have $Tu = 0$ and $Tv = 0$.

So we have

$$\begin{aligned} T(u+v) &= Tu + Tv \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore, $u+v \in \text{null } T$.

• closed under scalar multiplication

Suppose we have $u \in \text{null } T$ and $\lambda \in \mathbb{F}$. Then $Tu = 0$.

So we have

$$\begin{aligned} T(\lambda u) &= \lambda Tu \\ &= \lambda \cdot 0 \\ &= 0 \end{aligned}$$

Therefore, $\lambda u \in \text{null } T$.

So we conclude that null T is a subspace of V .

3.15 Definition:

A function $T: V \rightarrow W$ is called injective if $Tu = Tv$ implies $u = v$, for all $u, v \in V$.

3.16 Injectivity is equivalent to null space equals $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if $\text{null } T = \{0\}$.

Proof: Forward direction: If T is injective, then $\text{null } T = \{0\}$.

Suppose T is injective, since T is also linear, we have $T(0) = 0$, which means $0 \in \text{null } T$, and so $\{0\} \subset \text{null } T$.

We will prove $\text{null } T \subset \{0\}$. Suppose $v \in \text{null } T$. Then $Tv = 0$. But we also have $T(0) = 0$. Therefore, $Tv = T(0)$. ($Tv = 0 = T(0)$).

Since T is injective, $v = 0$. In other words, $v \in \{0\}$.

So $\text{null } T \subset \{0\}$. So we get the set equality $\text{null } T = \{0\}$.

Backward direction: If $\text{null } T = \{0\}$, then T is injective

Suppose $\text{null } T = \{0\}$. Suppose $u, v \in V$ satisfy $Tu = Tv$. Then we have

$$\begin{aligned} 0 &= Tu - Tv \\ &= T(u - v) \end{aligned}$$

Therefore, $u - v \in \text{null } T$. But $\text{null } T = \{0\}$. So $u - v \in \{0\}$, which implies $u - v = 0$, or equivalently $u = v$. So T is injective. \blacksquare

Range and surjectivity

3.17 Definition:

The range of a function $T: V \rightarrow W$ is a subset of W consisting of all vectors of the form Tv for some $v \in V$, and is denoted

$$\text{range } T = \{Tv : v \in V\}.$$

3.18 Example

Consider the zero map $0: V \rightarrow W$. Then $0v = 0$ for all $v \in V$. So we have

$$\begin{aligned}\text{range } 0 &= \{0v : v \in V\} \\ &= \{0 : v \in V\} \\ &= \{0\}\end{aligned}$$

Define $T = \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ by

$$T(x, y) = (2x, 5y, x+y)$$

Then we have

$$\begin{aligned}\text{range } T &= \{T(x, y) : (x, y) \in \mathbb{R}^2\} \\ &= \{(2x, 5y, x+y) : (x, y) \in \mathbb{R}^2\}\end{aligned}$$

A basis of range T is $(2, 0, 1), (0, 5, 1)$.

$$\begin{aligned}T(x, y) &= (2x, 5y, x+y) = (2x, 0, x) + (0, 5y, y) \\ &= x(2, 0, 1) + y(0, 5, 1)\end{aligned}$$

So $(2, 0, 1), (0, 5, 1)$ spans range T

Show that $(2, 0, 1), (0, 5, 1)$ is also linearly independent.

Then $(2, 0, 1), (0, 5, 1)$ is a basis of range T .

Let $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ be the differentiation map defined by $Dp = p'$.

For each polynomial $q \in \mathcal{P}(\mathbb{R})$, there exists a polynomial $p \in \mathcal{P}(\mathbb{R})$ that satisfies $p' = q$.

So we have

$$\begin{aligned} \text{Range } D &= \{ Dp : p \in \mathcal{P}(\mathbb{R}) \} \\ &= \{ p' : p \in \mathcal{P}(\mathbb{R}) \} \\ &= \{ q : q \in \mathcal{P}(\mathbb{R}) \} \\ &= \mathcal{P}(\mathbb{R}). \end{aligned}$$

3.19 The range is a subspace | If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W .

Proof: - Additive Identity

Suppose we have $T \in \mathcal{L}(V, W)$. Then by 3.11 of Axler, we have $T(0) = 0$. So $0 \in \text{range } T$.

• Closed under addition

Suppose $w_1, w_2 \in \text{range } T$. Then $w_1 = Tv_1$ and $w_2 = Tv_2$ for some $v_1, v_2 \in V$.

So we have

$$\begin{aligned} T(v_1 + v_2) &= Tv_1 + Tv_2 \\ &= w_1 + w_2 \end{aligned}$$

Since $v_1 + v_2 \in V$, it follows that we have $w_1 + w_2 \in \text{range } T$.

closed under multiplication

Suppose $\lambda \in F$ and $w \in \text{range } T$. Then $w = Tv$ for some $v \in V$. So we have

$$T(\lambda v) = \lambda Tv = \lambda w$$

since $\lambda v \in V$, it follows that we have $\lambda w \in \text{range } T$.
So $\text{range } T$ is a subspace of W .

3.22 Fundamental Theorem of Linear Maps

Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. Then $\text{range } T$ is finite-dimensional and

$$\dim V = \dim \text{null } T + \dim \text{range } T.$$

3.20 Definition:

A function $T: V \rightarrow W$ is surjective if its range equals W ; that is,

$$\text{range } T = W$$

Proof: let u_1, \dots, u_m be a basis of $\text{null } T$. then $\dim(\text{null } T) = m$. Also u_1, \dots, u_m is a linearly independent list. By 2.33 of Axler, we can extend this list to a basis

$$u_1, \dots, u_m, v_1, \dots, v_n \quad \text{of } V.$$

Then $\dim V = m+n$.

So we need to just show that range T is finite-dimensional, with $\dim(\text{range } T) = n$.

To accomplish this goal, we need to show that Tv_1, \dots, Tv_n is a basis of range T .

Let $v \in V$ be arbitrary. Since $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis of V , it spans V .

So we can write

$$v = a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_n v_n$$

for some $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$. So we have

$$Tv = T(a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_n v_n)$$

$$\left(\begin{array}{l} \text{additivity} \\ \text{of } T \end{array} \right) = T(a_1 u_1) + \dots + T(a_m u_m) + T(b_1 v_1) + \dots + T(b_n v_n)$$

$$\left(\begin{array}{l} \text{homogeneity} \\ \text{of } T \end{array} \right) = a_1 T u_1 + \dots + a_m T u_m + b_1 T v_1 + \dots + b_n T v_n$$

$$\left(\begin{array}{l} u_1, \dots, u_m \in \text{null } T \\ (u_1, \dots, u_m) \text{ is a basis} \\ \text{of null } T \end{array} \right) = a_1 \cdot 0 + \dots + a_m \cdot 0 + b_1 T v_1 + \dots + b_n T v_n \\ = b_1 T v_1 + \dots + b_n T v_n$$

Therefore, the list $(T v_1, \dots, T v_n)$ spans range T . Since we found a list that spans range T , we can conclude that range T is finite-dimensional.

Now we will show that $T v_1, \dots, T v_n$ is linearly independent. Suppose $c_1, \dots, c_n \in \mathbb{F}$ that satisfies

$$c_1 T v_1 + \dots + c_n T v_n = 0$$

Then we have

$$\begin{aligned}
 0 &= c_1 T v_1 + \dots + c_n T v_n \\
 &= T(c_1 v_1 + \dots + c_n v_n) \\
 &= T(c_1 v_1 + \dots + c_n v_n)
 \end{aligned}$$

Therefore, $c_1 v_1 + \dots + c_n v_n \in \text{null } T$.

Since u_1, \dots, u_m is a basis of $\text{null } T$, it spans $\text{null } T$, so we can write

$$\boxed{c_1 v_1 + \dots + c_n v_n} = d_1 u_1 + \dots + d_m u_m$$

for some $d_1, \dots, d_m \in \mathbb{F}$. So we get

$$-d_1 u_1 - \dots - d_m u_m + c_1 v_1 + \dots + c_n v_n = 0$$

But $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis of V , so it is linearly independent in V . So all scalars are zero:

$$-d_1 = 0, \dots, -d_m = 0, c_1 = 0, \dots, c_n = 0$$

In particular,

$$c_1 = 0, \dots, c_n = 0.$$

So $T v_1, \dots, T v_n$ is linearly independent.

So it is a basis of $\text{range } T$.

$$\text{So } \dim(\text{range } T) = n.$$

In summary, we have

$$\dim V = m + n$$

$$\dim(\text{null } T) = m$$

$$\dim(\text{range } T) = n$$

Therefore, $\dim V = m + n$

$$= \dim(\text{null } T) + \dim(\text{range } T)$$

as desired \square

3.23 A map to a smaller dimensional space is not injective

Suppose V and W are finite-dimensional vector spaces, that satisfy $\dim V > \dim W$. Then no linear map from V to W is injective.

Proof: Suppose by ~~contradiction~~ that we have $T \in \mathcal{L}(V, W)$.

Then, by the Fund. Thm of Linear Maps (3.22 of Axler), we have $\dim(\text{null } T) = \dim V - \dim(\text{range } T)$.
range T is a subspace of W \rightarrow $\dim(\text{range } T) \leq \dim W$
by 2.38 of Axler \rightarrow $\geq \dim V - \dim W$
 $\Rightarrow \dim(\text{range } T) \leq \dim W$ \rightarrow $> \dim W - \dim W = 0$.

So $\text{null } T \neq \{0\}$. By 3.16 of Axler, T is not injective. \square

3.24 A map to a larger dimensional space is not surjective

Suppose V and W are finite-dimensional vector spaces that satisfy $\dim V < \dim W$. Then no linear map from V to W is surjective.

Proof: let $T \in \mathcal{L}(V, W)$. Then by the Fund. Thm of Linear Maps (3.22 of Axler) we have

$0 \in \text{null } T$
 $\Rightarrow [0] \subset \text{null } T$
 $\Rightarrow \dim\{0\} \leq \dim \text{null } T$
 $0 \leq \dim \text{null } T$
 $(\dim \text{null } T \geq 0)$

$$\begin{aligned}
 \dim(\text{range } T) &= \dim V - \dim \text{null } T \\
 &\leq \dim V - 0 \\
 &= \dim V \\
 &< \dim W
 \end{aligned}$$

range $T \neq W$.

Therefore, T is not surjective.

Exercise 2. C1 of Axler,



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3.25 Example

Fix positive integers m, n . Consider the homogeneous system of linear equations

A system of m equations

$$\begin{cases}
 \sum_{k=1}^n A_{1,k} x_k = 0 \\
 \vdots \\
 \sum_{k=1}^n A_{m,k} x_k = 0
 \end{cases}$$

j th equation

$$\sum_{k=1}^n A_{j,k} x_k = 0$$

for some $A_{j,k} \in \mathbb{F}$, for $j = 1, \dots, m$ and $k = 1, \dots, n$.

Rephrase in terms of a linear map the question of whether this system of equations has a nonzero solution.

Proof: Define $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$ by

$$T(\underbrace{x_1, \dots, x_n}_{n \text{ coordinates}}) = \left(\underbrace{\sum_{k=1}^n A_{1,k} x_k, \dots, \sum_{k=1}^n A_{m,k} x_k}_{m \text{ coordinates}} \right)$$

Then the equation

$$T(\underbrace{(0, \dots, 0)}_n) = \underbrace{\left(\sum_{k=1}^n A_{1,k} x_k, \dots, \sum_{k=1}^n A_{m,k} x_k \right)}_m$$

is equivalent to the homogeneous system of equations.


Note that

$$(x_1, \dots, x_n) = (0, \dots, 0)$$

is a solution to the homogeneous system of equations. This is equivalent to saying

$$T(\underbrace{(0, \dots, 0)}_n) = \underbrace{(0, \dots, 0)}_m$$

$$\begin{aligned} x_1 &= 0 \\ &\vdots \\ x_n &= 0 \end{aligned}$$

There exists  nonzero solutions

x_1, \dots, x_n to the homogeneous system of equations

if and only if

$$\text{null } T \neq \{0\}.$$

In other words, if and only if there exist x_1, \dots, x_n , not all zero such that $(x_1, \dots, x_n) \in \text{null } T$.

3.26 Homogeneous system of linear equations

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Proof: Again, define $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$ by

$$T(\underbrace{(x_1, \dots, x_n)}_n) = \underbrace{\left(\sum_{k=1}^n A_{1,k} x_k, \dots, \sum_{k=1}^n A_{m,k} x_k \right)}_m$$

Then we have a homogeneous system of m linear equations with n variables x_1, \dots, x_n . Since there are more variables than there are equations, there

We have $\dim \mathbb{F}^n = n > m = \dim \mathbb{F}^m$

By 3.23 of Axler, $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$ is not injective
By 3.16 of Axler, null ~~space~~ $T \neq \underbrace{\{0, \dots, 0\}}_n$. So
there exists non zero solutions
of the system of equations.

Example

$$\begin{cases} \sum_{k=1}^n A_{1,k} x_k = 0 \\ \sum_{k=1}^n A_{2,k} x_k = 0 \end{cases}$$

And x_1, x_2, x_3 solve the
system of equations,
then at least one of x_1, x_2, x_3
is non zero.

3.27 Example

Rephrase in terms of a linear map the question of whether the
inhomogeneous system of linear equations

$$\begin{aligned} \sum_{k=1}^n A_{1,k} x_k &= c_1 \\ &\vdots \\ \sum_{k=1}^n A_{m,k} x_k &= c_m \end{aligned}$$

for any $A_{j,k} \in \mathbb{F}$. ($j = 1, \dots, m$ and $k = 1, \dots, n$) and for some $c_j, c_m \in \mathbb{F}$,
has no solution.

Define $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$ by

$$T(\underbrace{x_1, \dots, x_n}_n) = \left(\underbrace{\sum_{k=1}^n A_{1,k} x_k, \dots, \sum_{k=1}^n A_{m,k} x_k}_m \right)$$

Then the equation $T(x_1, \dots, x_n) = (c_1, \dots, c_m)$
is equivalent to the inhomogeneous system of
equations



3.29 Inhomogeneous system of linear equations

An inhomogeneous system of linear equations with more equations than the variables has no solution for some choice of constant terms.

Proof: Define $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$ by

$$T(x_1, \dots, x_n) = \left(\sum_{k=1}^n A_{1,k} x_k, \dots, \sum_{k=1}^n A_{m,k} x_k \right)$$

Since there are more equations than there are variables, we have

$$\dim \mathbb{F}^n = n < m = \dim \mathbb{F}^m$$

By 3.24 of Axler, $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$ is NOT surjective. In other words,

$$\text{range } T \neq \mathbb{F}^m$$

So there exist $c_1, \dots, c_m \in \mathbb{F}^m \setminus \text{range } T$ such that

$$T(x_1, \dots, x_n) \neq (c_1, \dots, c_m)$$

So we have

$$\sum_{k=1}^n A_{1,k} x_k \neq c_1$$

$$\sum_{k=1}^n A_{m,k} x_k \neq c_m,$$

which means the system of inhomogeneous equations with this choice of c_1, \dots, c_m does NOT contain any solutions.

