7/9/2019 **3B Null spaces and Ranges**

Defn 3.12 If we have $T \in \mathcal{L}(V,W)$, then the <u>null space</u> of T is the subset of V consisting of vectors in V that T maps

$$\operatorname{null} T = \{v \in V : Tv = 0\}$$

Eg 3.13 Consider the zero map $0 \in \mathcal{L}(V,W)$. For all $v \in V$, we have $0v = 0$.

Therefore, $\operatorname{null} 0 = \{v \in V : 0v = 0\}$
$= V$

· Define $4 \in \mathcal{L}(\mathbb{C}^3, \mathbb{C})$ by

$$4(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3.$$

Then we have

$$\operatorname{null} 4 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : 4(z_1, z_2, z_3) = 0\}$$
$$= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + 2z_2 + 3z_3 = 0\}$$

The basis of null 4 is $(-2,1,0), (-3,0,1)$

$$(z_1, z_2, z_3) = (-2z_2 - 3z_3, z_2, z_3)$$
$$= (-2z_2, z_2, 0) + (-3z_3, 0, z_3)$$
$$= z_2(-2,1,0) + z_3(-3,0,1)$$

so $(-2,1,0), (-3,0,1)$ spans null 4.

Then prove that $(-2,1,0), (-3,0,1)$ is linearly independent.

so $(-2,1,0), (-3,0,1)$ is a basis of null 4.

· Let $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ be the differentation map defined by $Dp = p'$

Then we have

$$\operatorname{null} D = \{p \in P(\mathbb{R}) : p'(z) = 0 \text{ for all } z \in \mathbb{F}\}$$
$$= \{p \in P(\mathbb{R}) : p'(z) = c \text{ for all } z \in \mathbb{F}, \text{ for some constant } c\}$$
$$= \{p \in P(\mathbb{R}) : p \text{ is a constant function}\}.$$

• Define $T \in \mathcal{L}\{P(\mathbb{R}), P(\mathbb{R})\}$ by $(Tp)(x) = x^2 p(x)$ for all $x \in \mathbb{R}$

Then we have

$$\begin{aligned} \text{null}\, T &= \{p \in P(\mathbb{R}) : Tp = 0\} \\ &= \{p \in P(\mathbb{R}) : (Tp)(x) = 0 \text{ for all } x \in \mathbb{R}\} \\ &= \{p \in P(\mathbb{R}) : x^2 p(x) = 0 \text{ for all } x \in \mathbb{R}\} \\ &= \{p \in P(\mathbb{R}) : p = 0\} \\ &= \{0\} \end{aligned}$$

The only polynomial that satisfies

$$x^2 p(x) = 0$$

is $p(x) = 0$

Defn 3.14 <u>Null space ~~is a subspace~~</u>

Suppose we have $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

proof:
• Additive Identity
Since we have $T \in \mathcal{L}(V, W)$, it follows that $T: V \to W$ is a linear map. By 3.11 of Axler, we have $T(0) = 0$.
Therefore, $0 \in \text{null}\, T$.

• Closed under addition
Suppose we have $u, v \in \text{null}\, T$. Then we have $Tu = 0$ and $Tv = 0$. So we have

$$\begin{aligned} T(u+v) &= Tu + Tv \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore, $u + v \in \text{null}\, T$.

• Closed Under Scalar multiplication
Suppose we have $u \in \text{null}\, T$ and $\lambda \in \mathbb{F}$. Then $Tu = 0$.
So we have

$$T(\lambda u) = \lambda Tu$$
$$= \lambda \cdot 0$$
$$= 0.$$

Therefore, $\lambda u \in \text{null}\,T$.

So we conclude that null T is a subspace of V. ☑

Defn 3.15 A function $T: V \to W$ is called <u>injective</u> if $Tu = Tv$ implies $u = v$, for all $u, v \in V$

Defn 3.16 <u>Injectivity is equivalent to null space equals $\{0\}$.</u>

"one-to-one" Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$.

proof: ① <u>Forward Direction: If T is injective, then null $T = \{0\}$</u>

Suppose T is injective. Since T is linear, we have $T(0) = 0$, which means $0 \in \text{null}\,T$, and so $\{0\} \subset \text{null}\,T$

We will prove null $T \subset \{0\}$.

Suppose $v \in \text{null}\,T$.

Then $Tv = 0$ But we also have $T(0) = 0$.

Therefore, $Tv = T(0)$, $(Tv = 0 = T(0))$

Since T is injective, $v = 0$. In other words, $v \in \{0\}$.

So null $T \subset \{0\}$. So we get the set equality null $T = \{0\}$

② <u>Backward Direction: If null $T = \{0\}$, then T is injective</u>

Suppose null $T = \{0\}$. Suppose $u, v \in V$ satisfy $Tu = Tv$. Then we have

$$0 = Tu - Tv$$
$$= T(u - v)$$

Therefore, $u - v \in \text{null}\,T$. But null $T = \{0\}$. So $u - v \in \{0\}$, which implies $u - v = 0$, or equivalently, $u = v$.

So T is injective.

☑

Range and Surjectivity

Defn 3.17 The *range* of a function $T: V \to W$ is a subset of W consisting of all vectors of the form Tv for some $v \in V$ and is denoted

$$\text{range } T = \{Tv : v \in V\}$$

Eg 3.18 Consider the zero map $0: V \to W$. Then $0v = 0$ for all $v \in V$. So we have

$$\begin{aligned}\text{range } 0 &= \{0v : v \in V\} \\ &= \{0 : v \in V\} \\ &= \{0\}\end{aligned}$$

- Define $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ by

$$T(x,y) = (2x, 5y, x+y).$$

Then we have

$$\begin{aligned}\text{range } T &= \{T(x,y) : (x,y) \in \mathbb{R}^2\} \\ &= \{(2x, 5y, x+y) : (x,y) \in \mathbb{R}^2\}\end{aligned}$$

A basis of range T is $(2,0,1), (0,5,1)$.

$$\begin{aligned}T(x,y) = (2x, 5y, x+y) &= (2x, 0, x) + (0, 5y, y) \\ &= x(2,0,1) + y(0,5,1).\end{aligned}$$

So $(2,0,1), (0,5,1)$ spans range T.
Show that $(2,0,1)(0,5,1)$ is also linearly independent.
Then $(2,0,1), (0,5,1)$ is a basis of range T.

- Let $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ be the differentiation map defined by

$$Dp = p'$$

For each polynomial $q \in \mathcal{P}(\mathbb{R})$, there exists a polynomial $p \in \mathcal{P}(\mathbb{R})$ that satisfies

$$p' = q.$$

So we have

$$\begin{aligned}\text{range } D &= \{Dp : p \in \mathcal{P}(\mathbb{R})\} \\ &= \{p' : p \in \mathcal{P}(\mathbb{R})\} \\ &= \{\}\end{aligned}$$

$= \{q : q \in P(\mathbb{R})\}$

$= P(\mathbb{R}).$

Defn 3.19 The range is a subspace. If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W.

proof: • Additive Identity

Suppose we have $T \in \mathcal{L}(V, W)$. Then by 3.11 of Axler, we have $T(0) = 0$. So $0 \in$ range T.

• Closed Under Addition

Suppose we have $w_1, w_2 \in$ range T. Then $w_1 = Tv_1$ and $w_2 = Tv_2$ for some $v_1, v_2, \in V$.

So we have

$$T(v_1 + v_2) = Tv_1 + Tv_2$$
$$= w_1 + w_2$$

Since $v_1 + v_2 \in V$, it follows that we have $w_1 + w_2 \in$ range T.

• Closed Under Multiplication

Suppose $\lambda \in \mathbb{F}$ and $w \in$ range T. Then $w = Tv$ for some $v \in V$.

So we have

$$T(\lambda v) = \lambda T v$$
$$= \lambda w$$

Since $\lambda v \in V$, it follows that we have $\lambda w \in$ range T

So range T is a subspace of W.

Thm 3.22 Fundamental Theorem of Linear Maps

Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Defn 3.20 A function $T : V \to W$ is surjective if its range equals W, that is

$$\operatorname{range} T = W$$

proof: Let $u_1, \dots, u_m$ be a basis of null T. Then $\dim(\text{null}\,T) = m$.
Also, $u_1, \dots, u_m$ is a linearly independent list.
By 2.33 of Axler, we can extend this list to a basis

$$u_1, \dots, u_m, v_1, \dots, v_n$$

of V. Then $\dim V = m + n$
So, we need to just show that range T is finite-dimensional with $\dim(\text{range}\,T) = n$. To accomplish this goal, we need to show that $Tv_1, \dots, Tv_n$ is a basis of range T.
Let $v \in V$ be arbitrary. Since $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis of V, it spans V.
So we can write

$$v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n$$

for some $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$. So we have

$$\begin{aligned}
Tv &= T(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) \\
\text{(additivity of } T) \quad &= T(a_1u_1) + \cdots + T(a_mv_m) + T(b_1v_1) + \cdots T(b_nv_n) \\
\text{(homogeneity of } T) \quad &= a_1Tu_1 + \cdots + a_mTv_m + b_1Tv_1 + \cdots + b_nTv_n \\
\left(\begin{array}{l} u_1, \dots, u_m \in \text{null}\,T \\ u_1, \dots, u_m \text{ is a basis of null}\,T \end{array}\right) \quad &= a_1 \cdot 0 + \cdots + a_m \cdot 0 + b_1Tv_1 + \cdots b_nTv_n \\
&= b_1Tv_1 + \cdots b_nTv_n
\end{aligned}$$

Therefore, the list $Tv_1, \dots, Tv_n$ spans range T.
Since we found a list that spans range T, we conclude that range T is finite-dimensional.
Now we will show that $Tv_1, \dots, Tv_n$ is linearly independent.
Suppose $c_1, \dots, c_n \in \mathbb{F}$ that satisfy

$$c_1Tv_1 + \cdots + c_nTv_n = 0$$

Then we have

$$\begin{aligned}
0 &= c_1Tv_1 + \cdots + c_nTv_n \\
&= T(c_1v_1) + \cdots + T(c_nv_n)
\end{aligned}$$

$= T(c_1 v_1 + \cdots + c_n v_n)$

Therefore, $c_1 v_1 + \cdots + c_n v_n \in \operatorname{null} T$.

Since $u_1, \cdots, u_m$ is a basis of null T, it spans null T, so we can write

$$\underbrace{\boxed{c_1 v_1 + \cdots + c_n v_n}}_{}^{\text{vector in null } T} = d_1 u_1 + \cdots + d_m u_m$$

for some $d_1, \cdots, d_m \in \mathbb{F}$. So we got

$-d_1 u_1 - \cdots - d_m u_m + c_1 v_1 + \cdots + c_n v_n = 0$

But $u_1, \cdots, u_m, v_1, \cdots, v_n$ is a basis of V, so it is linearly independent on V.

So all scalars are zero:

$-d_1 = 0, \cdots, -d_m = 0, c_1 = 0, \cdots, c_n = 0$.

In particular,

$c_1 = 0, \cdots, c_n = 0$.

So $Tv_1, \cdots, Tv_n$ is linearly independent.

So it is a basis of range T.

so $\dim(\operatorname{range} T) = n$.

In summarity, we have:

$\dim V = m + n$

$\dim(\operatorname{null} T) = m$

$\dim(\operatorname{range} T) = n$

Therefore,

$$\begin{aligned} \dim V &= m + n \\ &= \dim(\operatorname{null} T) + \dim(\operatorname{range} T) \end{aligned}$$

as desired. ∎

Def 3.23 A map to a smaller dimensional space is not injective

Suppose V and W are finite-dimensional vector spaces that satisfy $\dim V > \dim W$. Then no linear map from V to W is injective.

proof →

proof. Suppose that we have $T \in \mathcal{L}(V,W)$
Then, by the Fund. Thm. of Linear Maps (3.22 Axler)
we have
$$\begin{aligned}\dim(\operatorname{null} T) &= \dim V - \dim(\operatorname{range} T)\\ &\geq \dim V - \dim W\\ &> \dim W - \dim W\\ &= 0.\end{aligned}$$

range T is a subspace of W
by 2.38 of Axler
$\Rightarrow \dim(\operatorname{range} T) \leq \dim W$

So $\operatorname{null} T \neq \{0\}$. By 3.16 of Axler, T is not injective ∎

Defn 3.24 A map to a larger dimensional space is no subjective
Suppose V and W are finite-dimensional vector space that satisfy $\dim V < \dim W$. Then no linear map from V to W is subjective

Proof. Let $T \in \mathcal{L}(V,W)$. Then by the Fund. Thm of Linear Maps (3.22 of Axler), we have
$$\begin{aligned}\dim(\operatorname{range} T) &= \dim V - \dim \operatorname{null} T\\ &\leq \dim V - 0\\ &= \dim V\\ &< \dim W\end{aligned}$$

$0 \in \operatorname{null} T$
$\Rightarrow \{0\} \subset \operatorname{null} T$
by 2.38 of Axler $\Rightarrow \dim\{0\} \leq \dim \operatorname{null} T$
$0 \leq \dim \operatorname{null} T$
$(\dim \operatorname{null} T \geq 0)$

By Excercise 2.C1 of Axler, range $T \neq W$.

Therefore, T is not surjective ∎

7/10/2019

Eg 3.25 Fix positive integers m, n.
Consider the homogeneous system of linear equations

a system of m equations
$$\begin{cases}\sum_{k=1}^{n} A_{1,k}\, x_k = 0\\ \quad\vdots\\ \sum_{k=1}^{n} A_{m,k}\, x_k = 0\end{cases}$$

jth equation
$$\sum_{k=1}^{n} A_{j,k}\, x_k = 0$$

for some $A_{j,k} \in \mathbb{F}$, fro $j=1,\dots,m$ and for $k=1,\dots,n$
Rephrase in terms of a linear map the question of whether this system of equations has a nonzero solution

proof. Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by
$$T\underbrace{(x_1,\dots,x_n)}_{\text{coordinates}} = \underbrace{\left(\sum_{k=1}^{n} A_{1,k}x_k, \dots, \sum_{k=1}^{n} A_{m,k}x_k\right)}_{\text{coordinates}}$$

then the equation
$$T\underbrace{(0,\dots,0)}_{n} = \underbrace{(0,\dots,0)}_{m}$$
$\sum_{k=1}^{n} A_{1,k}x_k$ $\quad \sum_{k=1}^{n} A_{m,k}x_k$

is equivalent to the homogeneous system of equations ∎

Note that $(x_1,\dots,x_n) = (0,\dots,0)$ is a solution to the homogeneous system of equations
$x_1 = 0$
$\vdots$
$x_n = 0$

This is equivalent to saying
$$T\underbrace{(0,\dots,0)}_{n} = \underbrace{(0,\dots,0)}_{m}$$

There exist nonzero solutions $x_1,\dots,x_n$ to the homogeneous system of equations if and only if $\operatorname{null} T \neq \{0\}$
In other words, if and only if there exist $x_1,\dots,x_n$ not all zero such that $(x_1,\dots,x_n) \in \operatorname{null} T$.

Defn 3.26 Homogeneous system of linear equations

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Eg $\sum_{k=1}^{n} A_{1,k} x_k = 0$ and $\sum_{k=1}^{n} A_{2,k} x_k = 0$

and x_1, x_2, x_3 solve the above system of equations, then at least one of x_1, x_2, x_3 is nonzero.

proof: Again define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(\underbrace{x_1, \dots, x_n}_{n}) = \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{m,k} x_k \right)}$$

Then we have a homogeneous system of m linear equations with n variables $x_1, \dots, x_n$.

Since there are more variables than there are equations, we have

$$\dim \mathbb{F}^n = n > m = \dim \mathbb{F}^m.$$

By 3.23 of Axler, $T: \mathbb{F}^n \to \mathbb{F}^m$ is not injective

By 3.16 of Axler, $\operatorname{null} T \neq \{\underbrace{(0, \dots, 0)}\}$

So there exist nonzero solutions of the system of equations. ☑

Eg 3.27 Rephrase in terms of a linear map the question of whether the inhomogeneous system of linear equations

$$\sum_{k=1}^{n} A_{1,k} x_k = C_1$$

$$\vdots$$

$$\sum_{k=1}^{n} A_{m,k} x_k = C_m$$

for any $A_{j,k} \in \mathbb{F}$. ($j = 1, \dots, m$ and $k = 1, \dots, n$) and for some $C_1, \dots, C_m \in \mathbb{F}$, has no solution

Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(\underbrace{x_1, \dots, x_n}_{n \text{ coordinates}}) \quad \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{m,k} x_k \right)}_{m \text{ coordinates}}$$

Then the equation

$$T(x_1, \dots, x_n) = (c_1, \dots, c_m)$$

is equivalent to the inhomogeneous system of equations.

Defn 3.29 Inhomogeneous system of linear equations

An inhomogeneous system of linear equations with more equations than variables has no solution $\boxed{\text{for some}}$ choice of constant terms.

proof: Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(\underbrace{x_1, \dots, x_n}_{n}) = \underbrace{\left(\overset{c_1}{\sum_{k=1}^{n} A_{1,k} x_k}, \dots, \overset{c_m}{\sum_{k=1}^{n} A_{m,k} x_k} \right)}_{m}$$

Since there are more equations than there are variables we have

$$\dim \mathbb{F}^n = n < m = \dim \mathbb{F}^m$$

By 3.24 of Axler, $T: \mathbb{F}^n \to \mathbb{F}^m$ is NOT surjective

In other words,

$$\operatorname{range} T \neq \mathbb{F}^m$$

So there exist $c_1, \dots, c_m \in \mathbb{F}^m \setminus \operatorname{range} T$ such that

$$T(x_1, \dots, x_n) \neq (c_1, \dots, c_m)$$

So we have $\sum_{k=1}^{n} A_{1,k} x_k \neq c_1$

$$\vdots$$

$$\sum_{k=1}^{n} A_{m,k} x_k \neq c_m$$

→

which means the system of inhomogeneous equations with the choice of $C_1, \cdots, C_m$ does **NOT** contain any solutions. $\boxed{\exists}$