3.9 Algebraic Prop. of products of linear maps

• Associativity

If R, S, T are linear maps such that the product RST makes sense, then $(RS)T = R(ST)$

• Identity

If $T \in \mathcal{L}(V, W)$ & $I : V \to V$ is an Identity map, then $TI = T = IT$

• Distributive

If $T, T_1, T_2 \in \mathcal{L}(U, V)$ & $S, S_1, S_2 \in \mathcal{L}(V, W)$, then
$(S_1 + S_2)T = S_1T + S_2T$ & $S(T_1 + T_2) = ST_1 + ST_2$

3.11 Linear maps take 0 to 0

If $T \in \mathcal{L}(V, W)$, then $T(0) = 0$

Proof: since T is linear, we can use additivity to get

$$T(0) = T(0+0) = T(0) + T(0) = 2T(0)$$

Therefore $T(0) = 0$, as defined.

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3.B Null Spaces & Ranges

3.12 Definition

If we have $T \in \mathcal{L}(V, W)$, then the null space of T is the subset of V consist of vectors in V that T maps to 0: $\text{null } T = \{v \in V : Tv = 0\}$

3.13 example

• consider the zero map $0 \in \mathcal{L}(V, W)$. For all $v \in V$, we have $0v = 0$.
Therefore, $\text{null } 0 = \{v \in V : 0v = 0\} = V$

• Define $P \in \mathcal{L}(\mathbb{C}^3, \mathbb{C})$ by $P(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$
Then we have

$$\text{null } P = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : P(z_1, z_2, z_3) = 0\} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + 2z_2 + 3z_3 = 0\}$$

The basis of null P is $(-2, 1, 0), (-3, 0, 1)$

$$\begin{aligned}(z_1, z_2, z_3) &= (-2z_2 - 3z_3, z_2, z_3) \\ &= (-2z_2, z_2, 0) + (-3z_3, 0, z_3) \\ &= z_2(-2, 1, 0) + z_3(-3, 0, 1)\end{aligned}$$

so $(-2,1,0), (-3,0,1)$ spans null P. Then prove that $(-2,1,0)$, $(-3,0,1)$ is linearly independent. So $(-2,1,0), (-3,0,1)$ is a basis of null P.

- Let $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ be the differentiation map defined by $Dp = p'$. Then we have,

$$\begin{aligned}\text{null } D &= \{p \in P(\mathbb{R}) : p'(z) = 0 \text{ for all } z \in \mathbb{F}\} \\ &= \{p \in P(\mathbb{R}) : p'(z) = c \text{ for all } z \in \mathbb{F}, \text{ for some constant } c\} \\ &= \{p \in P(\mathbb{R}) : p \text{ is a constant function}\}\end{aligned}$$

- Define $T \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ by $(Tp)(x) = x^2 p(x)$ for all $x \in \mathbb{R}$. Then we have

$$\begin{aligned}\text{null } T &= \{p \in P(\mathbb{R}) : Tp = 0\} \\ &= \{p \in P(\mathbb{R}) : (Tp)(x) = 0 \text{ for all } x \in \mathbb{R}\} \\ &= \{p \in P(\mathbb{R}) : x^2 p(x) = 0 \text{ for all } x \in \mathbb{R}\} \\ &= \{p \in P(\mathbb{R}) : p(x) = 0 \text{ for all } x \in \mathbb{R}\} \\ &= \{p \in P(\mathbb{R}) : p = 0\} \\ &= \{0\}\end{aligned}$$

3.14 Null Space is a Subspace

Suppose we have $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

Proof:
- Additive Id
  Since we have $T \in \mathcal{L}(V, W)$, it follows that $T: V \to W$ is a linear map. By 3.11 of Axler, we have $T(0) = 0$. Therefore, $0 \in$ null T

- Closed under addition
  Suppose we have $u, v \in$ null T. Then we have $Tu = 0$ & $Tv = 0$
  so we have
  $$\begin{aligned}T(u+v) &= Tu + Tv \\ &= 0 + 0 \\ &= 0\end{aligned}$$
  Therefore, $u + v \in$ null T

- closed under scalar multiplication
  Suppose we have $u \in$ null T & $\lambda \in \mathbb{F}$. Then $Tu = 0$. So we have
  $$\begin{aligned}T(\lambda u) &= \lambda(Tu) \\ &= \lambda \cdot 0 \\ &= 0\end{aligned}$$
  Therefore, $\lambda u \in$ null T

So we conclude that null T is a subspace of V.

3.15 Defin.

A function $T: V \to W$ is called injective if $Tu = Tv$ implies $u = v$, for all $u, v \in V$.

3.16 Injectivity is Equivalent to null space equals $\{0\}$.

Let $T \in \mathcal{L}(V, W)$. Then T is injective if & only if null $T = \{0\}$.

Proof: Forward Direction: If T is injective, then null $T = \{0\}$.

Suppose T is injective. Since T is also linear, we have $T(0) = 0$, which means $0 \in$ null T, & so $\{0\} \subset$ null T.

We will prove null $T \subset \{0\}$. Suppose $v \in$ null T.

Then $Tv = 0$. But we also have $T(0) = 0$. Therefore, $Tv = T(0)$, $(Tv = 0 = T(0))$. Since T is injective, $v = 0$.

In other words, $v \in \{0\}$. So null $T \subset \{0\}$. So we get the set equality null $T = \{0\}$.

Backward Direction: If null $T = \{0\}$, then T is injective

Suppose null $T = \{0\}$. Suppose $u, v \in V$ satisfy $Tu = Tv$. Then we h

$0 = Tu - Tv$

$= T(u-v)$ Therefore, $u - v \in$ null T. But null $T = \{0\}$. So $u - v \in \{0\}$, which implies $u - v = 0$, or equivalently, $u = v$. So T injective

Range & Surjectivity

3.17 Def.

The range of a function $T: V \to W$ is a subset of W consisting of all vectors of the form Tv for some $v \in V$ & is denoted

$$\text{range } T = \{Tv : v \in V\}$$

3.18 Example

- Consider the zero map $0: V \to W$. Then $0v = 0$ for all $v \in V$. So we have

$$\begin{aligned}\text{range } 0 &= \{0v : v \in V\} \\ &= \{0 : v \in V\} \\ &= \{0\}\end{aligned}$$

- Define $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ by $T(x,y) = (2x, 5y, x+y)$

Then we have

$$\begin{aligned}\text{range } T &= \{T(x,y) : (x,y) \in \mathbb{R}^2\} \\ &= \{(2x, 5y, x+y) : (x,y) \in \mathbb{R}^2\}\end{aligned}$$

A basis of range T is $(2,0,1), (0,5,1)$

$$\begin{aligned}T(x,y) = (2x, 5y, x+y) &= (2x, 0, x) + (0, 5y, y) \\ &= x(2,0,1) + y(0,5,1)\end{aligned}$$

So $(2,0,1), (0,5,1)$ spans range T. Show that $(2,0,1), (0,5,1)$ is also linearly indep. Then $(2,0,1), (0,5,1)$ is a basis of range T.

• Let $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ be the differentiation map defined by $Dp = p'$.
For each polynomial $q \in P(\mathbb{R})$, there exists a polynomial $p \in P(\mathbb{R})$ that satisfies $p' = q$. So we have range $D = \{Dp : p \in P(\mathbb{R})\}$
$= \{p' = p \in P(\mathbb{R})\}$
$= \{q : q \in P(\mathbb{R})\}$
$= P(\mathbb{R})$.

3.19 The Range is a Subspace: If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W.

Proof: • additive identity

Suppose we have $T \in \mathcal{L}(V, W)$. Then, by 3.11 of Axler, we have $T(0) = 0$. So $0 \in$ range T.

• closed under add.

Suppose $w_1, w_2 \in$ range T. Then $w_1 = Tv_1$ & $w_2 = Tv_2$ for some $v_1, v_2 \in V$. So we have
$$T(v_1 + v_2) = Tv_1 + Tv_2$$
$$= w_1 + w_2$$
Since $v_1 + v_2 \in V$, it follows that we have $w_1 + w_2 \in$ range T

• closed under scalar mult.

Suppose $\lambda \in \mathbb{F}$ & $w \in$ range T. Then $w = Tv$ for some $v \in V$. So we have $T(\lambda v) = \lambda Tv$
$= \lambda w$ since $\lambda v \in V$, it follows that we have $\lambda w \in$ range T. So range T is a subspace of W.

3.22 Fundamental Thm of Linear Maps

Suppose V is a finite-dimensional vector space & $T \in \mathcal{L}(V, W)$. Then range T is finite-dim & $\dim V = \dim \text{null}\, T + \dim \text{range}\, T$

3.20 Def

A function $T: V \to W$ is surjective if its range equals W; that is, range $T =$

3.22 proof:

Let $u_1, \ldots, u_m$ be a basis of null T. Then $\dim(\text{null}\, T) = m$. Also, $u_1, \ldots, u_m$ is a linearly indep. list. By 2.33 of Axler, we can extend this list to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of V. Then, $\dim V = m + n$. So we need to just show that range T is finite-dimensional with $\dim(\text{range}\, T) = n$. To accomplish the goal, we need to show that $Tv_1, \ldots, Tv_n$ is a basis of range T.

Let $v \in V$ be arbitrary. Since $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V, it spans V. So we can write $v = a_1 u_1 + \ldots + a_m u_m + b_1 v_1 + \ldots + b_n v_n$ for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$. So we have $Tv = T(a_1 u_1 + \ldots + a_m u_m + b_1 v_1 + \ldots + b_n v$

(additivity of T) $= T(a_1u_1) + \dots + T(a_mu_m) + T(b_1v_1) + \dots + T(b_nv_n)$
(homogeneity of T) $= a_1Tu_1 + \dots + a_mTu_m + b_1Tv_1 + \dots + b_nTv_n$
($u_1, \dots, u_m \in$ null T
$u_1, \dots, u_m$ is a basis of null T) $= a_1 0 + \dots + a_m 0 + b_1Tv_1 + \dots + b_nTv_n$
$= b_1Tv_1 + \dots + b_nTv_n$

Therefore, the list $Tv_1, \dots, Tv_n$ spans range T. Since we found a list that spans range T, we conclude that range T is finite-dim

Now we will show that $Tv_1, \dots, Tv_n$ is lin. indep.
Suppose $c_1, \dots, c_n \in F$ that satisfy
$$c_1Tv_1 + \dots + c_nTv_n = 0$$
Then we have
$$0 = c_1Tv_1 + \dots + c_nTv_n$$
$$= T(c_1v_1) + \dots + T(c_nv_n)$$
$$= T(c_1v_1 + \dots + c_nv_n)$$
Therefore, $c_1v_1 + \dots + c_nv_n \in$ null T
Since $u_1, \dots, u_m$ is a basis of null T, it spans null T, so we can write $c_1v_1 + \dots + c_nv_n = d_1u_1 + \dots + d_mu_m$ for some $d_1, \dots, d_m \in F$
So we get $-d_1u_1 - \dots - d_mu_m + c_1v_1 + \dots + c_nv_n = 0$
But $u_1, \dots, u_m, v_1, \dots, v_n$ is also a basis of V.
So all scalars are zero: $-d_1 = 0, \dots, -d_m = 0, c_1 = 0, \dots, c_n =$
In particular
$c_1 = 0, \dots, c_n = 0$ so $Tv_1, \dots, Tv_n$ is lin indep.
So it is a basis of range T, so $\dim(\text{range } T) = n$.

In summary, we have:
$\dim V = m+n$
$\dim(\text{null } T) = m$
$\dim(\text{range } T) = n$
Therefore, $\dim V = m+n$
$= \dim(\text{null } T) + \dim(\text{range } T)$, as desired.

3.23 <u>A map to a smaller dim space is not injective</u>
Suppose V & W are finite-dim vector spaces that satisfy $\dim V >$ $\dim W$. Then no linear map V to W is injective

<u>proof:</u> suppose that we have $T \in \mathcal{L}(V, W)$. Then by the fund Thm of lin. Maps (3.22 of Axler), we have $\dim(\text{null } T)$
$= \dim V - \dim(\text{range } T)$
$\geq \dim V - \dim W$
$> \dim W - \dim W$
$= 0$

range T is a subspace of W
by 2.38 of Axler $\dim(\text{range } T) \leq \dim W$

So null $T \neq \{0\}$. By 3.16 of Axler, T is not injective

3.24 A map to a larger dim space is not surjective

Suppose V & W are finite dim. vector spaces that satisfy $\dim V < \dim W$. Then no lin. map from V to W is surjective.

Proof: Let $T \in \mathcal{L}(V, W)$. Then by the fund. Thm of Lin Maps (3.22 Axler) we have
$$\dim(\text{range } T) = \dim V - \dim \text{null } T$$
$$\leq \dim V - 0$$
$$= \dim V$$
$$< \dim W$$

$0 \in \text{null } T$

by 2.38 of Axler: $\Rightarrow \{0\} \subset \text{null } T$

$\dim \{0\} \leq \dim \text{null } T$

$0 \leq \dim \text{null } T$

$(\dim \text{null } T \geq 0)$

By exercise 2.C1 of Axler, range $T \neq W$. Therefore, T is not surjective

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3.25 Example fix positive integers m, n.

Consider the homogeneous sys. of lin eqths

a system of m equations
$$\begin{cases} \sum_{k=1}^{n} A_{1,k} x_k = 0 \\ \vdots \\ \sum_{k=1}^{n} A_{m,k} x_k = 0 \end{cases}$$

for some $A_{j,k} \in \mathbb{F}$, for $j = 1, \ldots, m$ & for $k = 1, \ldots, n$.

Rephrase in terms of a linear map the question of whether this system of eqths has a nonzero soln

Proof: Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by $T\underbrace{(x_1, \ldots, x_n)}_{n \text{ coordinates}} = \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k\right)}_{m \text{ coor}}$.

$\sum_{k=1}^{n} A_{1,k} x_k$

Then the eqth $T\underbrace{(0, \ldots, 0)}_{n} = \underbrace{(0, \ldots, 0)}_{m}$ is equivalent to the homogeneous sys of eqths

$\sum_{k=1}^{n} A_{m,k} x_k$

Note that $(x_1, \ldots, x_n) = (0, \ldots, 0)$ is $x_1 = 0, \ldots, x_n = 0$

a soln to the homogeneous system of eqths. This is equivalent to saying $T\underbrace{(0, \ldots, 0)}_{n} = \underbrace{(0, \ldots, 0)}_{m}$

There exist nonzero solns $x_1, \ldots, x_n$ to the homogeneous system of eqths if & only if null $T \neq \{0\}$. In other words, if & only if there exist $x_1, \ldots, x_n$ not all zero such that $(x_1, \ldots, x_n) \in \text{null } T$

3.26 Homogeneous system of Lin Eqths

A homogeneous system of linear eqths w/ more variables than eqths has nonzero solns.

Example
$$\sum_{k=1}^{n} A_{1,k} x_k = 0$$
$$\sum_{k=1}^{n} A_{2,k} x_k = 0$$

Proof: Again, define $T: \mathbb{F}^n \to \mathbb{F}^m$ by $T(x_1, \ldots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k\right)$

Then we have a homogeneous sys of m linear eqtns w/n variables $x_1, \dots, x_n$. Since there are more variables then there are eqtns, we have

$$\dim \mathbb{F}^n = n > m = \dim \mathbb{F}^m$$

By 3.23 of Axler, $T: \mathbb{F}^n \to \mathbb{F}^m$ is not injective.

By 3.16 of Axler, null $T \neq \{\underbrace{0, \dots, 0}_{n}\}$

And x_1, x_2, x_3 solve the above system of eqtns. Then at least one of x_1, x_2, x_3 is nonzero.

3.22 Example

Rephrase in terms of a lin. map the question of whether the inhomogeneous system of linear eqtns

$$\sum_{k=1}^{n} A_{1,k} x_k = c_1$$
$$\vdots$$
$$\sum_{k=1}^{n} A_{m,k} x_k = c_m, \text{ for any } A_{j,k} \in \mathbb{F} \ (j = 1, \dots, m \text{ \& } k = 1, \dots$$

& for some $c_1, \dots, c_m \in \mathbb{F}$, has no soln.

Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by $T(x_1, \dots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{m,k} x_k\right)$

Then the $T(x_1, \dots, x_n) = (c_1, \dots, c_n)$ equation is equivalent to the inhomogeneous sys. of eqtns.

3.29 Inhomogeneous system of linear Eqtns

An inhomogeneous system of linear eqtns with more eqtns than the variables have no soln for some choice of consistent terms.

Proof: Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by $T(\underbrace{x_1, \dots, x_n}_{n}) = \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{m,k} x_k\right)}_{m}$

Since there are more eqtns than there are variables, $m > n$, we have

$$\dim \mathbb{F}^n = n < m = \dim \mathbb{F}^m$$

By 3.24 of Axler, $T: \mathbb{F}^n \to \mathbb{F}^m$ is not surjective. In other words, range $T \neq \mathbb{F}^m$. So there exist $c_1, \dots, c_m \in \mathbb{F}^m \setminus$ range T such that

$$T(x_1, \dots, x_n) \neq (c_1, \dots, c_m)$$

So we have

$$\sum_{k=1}^{n} A_{1,k} x_k \neq c_1$$
$$\vdots$$
$$\sum_{k=1}^{n} A_{m,k} x_k \neq c_m,$$

which means the system of inhomogeneous eqtns w/ this choice of $c_1, \dots, c_m$ does not

3.C Matrices

3.30 Def.

Let m & n denote positive integers

An $m \times n$ matrix A is a rectangular array of elements of $\mathbb{F}$, w/ m rows & n columns:

$$\begin{array}{c} \\ A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix} \begin{array}{l} \\ \\ j^{th} \text{ row} \\ \\ \\ \end{array} \\ k^{th} \text{ column} \end{array}$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A

row # column #

3.31 Example

If $A = \begin{pmatrix} 8 & 4 & 5-3i \\ 1 & 9 & 7 \end{pmatrix}$, then $A_{1,1} = 8$ $A_{1,2} = 4$ $A_{1,3} = 5-3i$

$A_{2,1} = 1$ $A_{2,2} = 9$ $A_{2,3} = 7$

3.32 Def.

Suppose $T \in \mathcal{L}(V,W)$ & $v_1, \ldots, v_n$ is a basis of V & $w_1, \ldots, w_m$ is a basis of W. The matrix of T w/ respect to these bases is the $m \times n$ matrix

$$M(T(v_1, \ldots, v_n), (w_1, \ldots, w_m))$$

whose entries $A_{j,k}$ are defined by $Tv_k = A_{1,k} w_1 + \ldots + A_{m,k} w_m$

If the basis are understood, then we can just write $M(T)$ to denote the matrix of T w/ respect to the bases.

$$M(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix}$$

For any $k = 1, \ldots, n$, we have

$$Tv_k = \underbrace{(w_1, \ldots, w_m)}_{1 \times m} \underbrace{\begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}}_{m \times 1}$$

$$= \underbrace{(A_{1,k} w_1 + \ldots + A_{m,k} w_m)}_{1 \times 1}$$

Identify all 1×1 matrices by their one entry

$$= A_{1,k} w_1 + \ldots + A_{m,k} w_m$$

$$= \sum_{j=1}^{m} A_{j,k} w_j$$

<u>3.33 Example</u> Define $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ by $T(x,y) = (x+3y,\ 2x+5y,\ 7x+9y)$

Find the matrix of T w/ respect to the standard bases of $\mathbb{F}^2$; $\mathbb{F}^3$.

In other words, find

$$M(T) = M\Big(T, \underbrace{(\overset{v_1}{(1,0)}, \overset{v_2}{(0,1)})}_{\text{standard basis of } \mathbb{F}^2}, \underbrace{(\overset{w_1}{(1,0,0)}, \overset{w_2}{(0,1,0)}, \overset{w_3}{(0,0,1)})}_{\text{standard basis of } \mathbb{F}^3}\Big)$$

Soln: we have

$$T(\overset{v_1}{1,0}) = (1,2,7)$$
$$T(\overset{v_2}{0,1}) = (3,5,9)$$

The matrix of T w/ respect to the standard bases is

$$M(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix} = \big(T(1,0)\ \ T(0,1)\big)$$

$$\begin{aligned} T(1,0) &= Tv_1 \\ &= \sum_{j=1}^{3} A_{j,1} w_j = A_{1,1}w_1 + A_{2,1}w_2 + A_{3,1}w_3 = 1(1,0,0) + \\ &\quad 2(0,1,0) + 7(0,0,1) = (1,2,7) \end{aligned}$$

$$\begin{aligned} T(0,1) &= Tv_2 \\ &= \sum_{j=1}^{3} A_{j,2} w_j = A_{1,2}w_1 + A_{2,2}w_2 + A_{3,2}w_3 \\ &= 3(1,0,0) + 5(0,1,0) + 9(0,0,1) \\ &= (3,5,9) \end{aligned}$$

<u>3.39 Example</u>

Let $1, x, x^2, \ldots, x^m$ be the standard basis of $P_m(\mathbb{F})$

Suppose $D \in \mathcal{L}(P_3(\mathbb{R}), P_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find the matrix of D w/ respect to the standard basis of $P_3(\mathbb{R})$ & $P_2(\mathbb{R})$. In other words, find

$$M(D) = M\Big(D, \underbrace{(1, x, x^2, x^3)}_{\text{standard basis of } P_3(\mathbb{R})}, \underbrace{(1, x, x^2)}_{\text{S.B. of } P_2(\mathbb{R})}\Big)$$

<u>Soln</u>: Basis of $P_3(\mathbb{R})$ is $1, x, x^2, x^3$

Basis of $P_2(\mathbb{R})$ is $1, x, x^2$

for all positive integers n), $D(x^n) = (x^n)' = nx^{n-1}$

Input in $P_3(\mathbb{R})$		corresponding vector in $\mathbb{F}^4$	output in $P_2(\mathbb{R})$		corr. vector in $\mathbb{F}^3$
1	$\longrightarrow$	$(1,0,0,0)$	$D(1) = 0$	$\longleftrightarrow$	$(0,0,0)$
x	$\longrightarrow$	$(0,1,0,0)$	$D(x) = 1$	$\longleftrightarrow$	$(1,0,0)$
x^2	$\longleftrightarrow$	$(0,0,1,0)$	$D(x^2) = 2x$	$\longleftrightarrow$	$(0,2,0)$
x^3	$\longleftrightarrow$	$(0,0,0,1)$	$D(x^3) = 3x^2$	$\longleftrightarrow$	$(0,0,3)$

$$M(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Input in $P_3(\mathbb{R})$		output in $P_2(\mathbb{R})$	verify w/th differentia map
1	$M(D)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix} 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&3 \end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$	0	$D(1)=0$
x	$M(D)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix} 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&3 \end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$	1	$D(x)=1$
x^2	$M(D)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix} 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&3 \end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix} = 2\begin{pmatrix}0\\1\\0\end{pmatrix}$	$2x$	$D(x^2)=2x$
x^3	$M(D)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix} 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&3 \end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\3\end{pmatrix} = 3\begin{pmatrix}0\\0\\1\end{pmatrix}$	$3x^2$	$D(x^3)=3x^2$