LECTURE 09

07-09-19

3.B Null Spaces and Ranges

If we have $T \in \mathcal{L}(V, W)$, then the null space of T is the subset of V consisting of vectors in V that T maps to 0:

$$\text{null } T = \{v \in V : Tv = 0\}$$

3.13 Example

- Consider the zero map $0 \in \mathcal{L}(V, W)$. For all $v \in V$, we have

$$0v = 0$$

Therefore,

$$\text{null } 0 = \{v \in V : 0v = 0\}$$
$$= V$$

- Define $\varphi \in \mathcal{L}(\mathbb{C}^3, \mathbb{C})$ by

$$\varphi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$$

Then we have

$$\text{null } \varphi = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \varphi(z_1, z_2, z_3) = 0\}$$
$$= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + 2z_2 + 3z_3 = 0\}$$

$$(z_1, z_2, z_3) = (-2z_2 - 3z_3, z_2, z_3)$$
$$= (-2z_2, z_2, 0) + (-3z_3, 0, z_3)$$
$$= z_2(-2, 1, 0) + z_3(-3, 0, 1)$$

so $(-2,1,0), (-3,0,1)$ spans null φ

Then prove that $(-2,1,0), (-3,0,1)$ is linearly independent.

So $(-2,1,0), (-3,0,1)$ is a basis of null φ

- Let $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ be the differentiation map defined by

$$Dp = p'$$

Then we have

$$\text{null } D = \{p \in P(\mathbb{R}) : p'(z) = 0 \text{ for all } z \in \mathbb{F}\}$$
$$= \{p \in P(\mathbb{R}) : p'(z) = c \text{ for all } z \in \mathbb{F}, \text{ for some constant } c\}$$
$$= \{p \in P(\mathbb{R}) : p \text{ is a constant function}\}$$

- Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ by

$$(Tp)(x) = x^2 p(x)$$

for all $x \in \mathbb{R}$

Then we have

$$\begin{aligned}
\text{null } T &= \{ p \in \mathcal{P}(\mathbb{R}) : Tp = 0 \} \\
&= \{ p \in \mathcal{P}(\mathbb{R}) : (Tp)(x) = 0 \text{ for all } x \in \mathbb{R} \} \\
&= \{ p \in \mathcal{P}(\mathbb{R}) : x^2 p(x) = 0 \text{ for all } x \in \mathbb{R} \} \\
&= \{ p \in \mathcal{P}(\mathbb{R}) : p(x) = 0 \text{ for all } x \in \mathbb{R} \} \\
&= \{ p \in \mathcal{P}(\mathbb{R}) : p = 0 \} \\
&= \{0\}
\end{aligned}$$

The only polynomial that satisfies

$$x^2 p(x) = 0$$

is $p = 0$

3.14 Null Space is a subspace

Suppose we have $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V

Proof:

- Additive identity

  Since we have $T \in \mathcal{L}(V, W)$ it follows that $T : V \to W$ is a linear map. By 3.11 of Axler, we have $T(0) = 0$

  Therefore, $0 \in$ null T

- Closed under addition

  Suppose we have $u, v \in$ null T. Then we have $Tu = 0$ and $Tv = 0$

  So we have

  $$\begin{aligned}
  T(u+v) &= Tu + Tv \\
  &= 0 + 0 \\
  &= 0
  \end{aligned}$$

  Therfore, $u + v \in$ null T

- Closed under scalar multiplication

  Suppose we have $u \in$ null T and $\lambda \in \mathbb{F}$. Then $Tu = 0$

  So we have

  $$\begin{aligned}
  T(\lambda u) &= \lambda T u \\
  &= \lambda \cdot 0 \\
  &= 0
  \end{aligned}$$

  Therefore, $\lambda u \in$ null T

So we conclude that null T is a subspace of V

3.15 Definition

A function $T : V \to W$ is called injective if $Tu = Tv$ implies $u = v$ for all $u, v \in V$.

3.16 Injectivity is equivalent to null space equals $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$

Proof: Forward direction: If T is injective, then null $T = \{0\}$

Suppose T is injective. Since T is also linear, we have $T(0) = 0$ which means $0 \in$ null T, and so $\{0\} \subset$ null T

We will prove null $T \subset \{0\}$. Suppose $v \in$ null T.

Then $Tv = 0$. But we also have $T(0) = 0$

Therefore, $Tv = T(0)$, $(Tv = 0 = T(0))$

Since T is injective, $v = 0$. In other words, $v \in \{0\}$

So null $T \subset \{0\}$. So we get the set equality null $T = \{0\}$

Backward direction: If null $T = \{0\}$, then T is injective

Suppose null $T = \{0\}$. Suppose $u, v \in V$ satisfy $Tu = Tv$

Then we have

$$\begin{aligned} 0 &= Tu - Tv \\ &= T(u - v) \end{aligned}$$

Therefore, $u - v \in$ null T. But null $T = \{0\}$. So $u - v \in \{0\}$, which implies $u - v = 0$, or equivalently, $u = v$.

So T is injective.

Range and Subjectivity

3.17 Definition

The range of a function $T : V \to W$ is a subset of W consisting of all vectors of the form Tv for som $v \in V$ and is denoted

$$\text{range } T = \{Tv : v \in V\}$$

3.18 Example

- Consider the zero map $0: V \to W$. Then $0v = 0$ for all $v \in V$

So we have

$$\begin{aligned} \text{range } 0 &= \{0v : v \in V\} \\ &= \{0 : v \in V\} \\ &= \{0\} \end{aligned}$$

- Define $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ by

$$T(x,y) = (2x, 5y, x+y)$$

Then we have

$$\begin{aligned} \text{range } T &= \{T(x,y) : (x,y) \in \mathbb{R}^2\} \\ &= \{(2x, 5y, x+y) : (x,y) \in \mathbb{R}^2\} \end{aligned}$$

A basis of range T is $(2,0,1), (0,5,1)$.

$$\begin{aligned} T(x,y) = (2x, 5y, x+y) &= (2x, 0, x) + (0, 5y, y) \\ &= x(2,0,1) + y(0,5,1) \end{aligned}$$

So $(2,0,1), (0,5,1)$ spans range T.

Show that $(2,0,1), (0,5,1)$ is also linearly independent

Then $(2,0,1), (0,5,1)$ is a basis of range T

- Let $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ be the differentiation map defined by

$$Dp = p'$$

For each polynomial $q \in \mathcal{P}(\mathbb{R})$, there exists a polynomial $p \in \mathcal{P}(\mathbb{R})$ that satisfies $p' = q$

So we have

$$\begin{aligned} \text{range } D &= \{Dp : p \in \mathcal{P}(\mathbb{R})\} \\ &= \{p' : p \in \mathcal{P}(\mathbb{R})\} \\ &= \{q : q \in \mathcal{P}(\mathbb{R})\} \\ &= \mathcal{P}(\mathbb{R}) \end{aligned}$$

3.19 The range is a subspace: If $T \in \mathcal{L}(V,W)$, then range T is a subspace of W

Proof:
- Additive identity

  Suppose we have $T \in \mathcal{L}(V,W)$. Then by 3.11 of Axler, we have $T(0) = 0$. So $0 \in$ range T

- Closed under addition

  Suppose $w_1, w_2 \in$ range T. then $w_1 = Tv_1$ and $w_2 = Tv_2$ for some $v_1, v_2 \in V$

  So we have

$$T(v_1 + v_2) = Tv_1 + Tv_2$$
$$= w_1 + w_2$$

  Since $v_1 + v_2 \in V$, it follows that we have $w_1 + w_2 \in$ range T

- Closed under multiplication

  Suppose $\lambda \in \mathbb{F}$ and $w \in$ range T. Then $w = Tv_1$ for some $v \in V$

  So we have

$$T(\lambda v) = \lambda Tv$$
$$= \lambda w$$

  Since $\lambda v \in V$, it follows that we have $\lambda w \in$ range T

So range T is a subspace of W

3.22 Fundamental Theorem of Linear Maps

Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V,W)$

Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

3.20 Definition

A function $T: V \to W$ is surjective if its range equals W; that is,

$$\operatorname{range} T = W$$

Proof: Let $u_1, \dots, u_m$ be a basis of null T. The $\dim(\operatorname{null} T) = m$

Also, $u_1, \dots, u_m$ is a linearly independent list. By 2.33 of Axler, we can extend the list to a basis

$$u_1, \dots, u_m, v_1, \dots, v_n$$

of V. Then $\dim V = m + n$

So we need to just show that range T is finite-dimensional with $\dim(\operatorname{range} T) = n$

To accomplish this goal, we need to show that $Tv_1, \dots, Tv_n$ is a basis of range T.

Let $v \in V$ be arbitrary. Since $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis of V, it spans V. So we can write

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$$

for some $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$. So we have

$$Tv = T(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n)$$

$$\left(\begin{array}{c}\text{additivity}\\ \text{of}\\ T\end{array}\right) = T(a_1 u_1) + \cdots + T(a_m u_m) + T(b_1 v_1) + \cdots + T(b_n v_n)$$

$$\left(\begin{array}{c}\text{homogeneity}\\ \text{of}\\ T\end{array}\right) = a_1 Tu_1 + \cdots + a_m Tu_m + b_1 Tv_1 + \cdots + b_n Tv_n$$

$$\left(\begin{array}{c}u_1, \dots, u_m \in \text{null } T\\ u_1, \dots, u_m \text{ is a basis of null } T\end{array}\right) = a_1 \cdot 0, \dots a_m \cdot 0, b_1 Tv_1 + \cdots + b_n Tv_n$$

$$= b_1 Tv_1 + \cdots + b_n Tv_n$$

Therefore, the list $Tv_1, \dots, Tv_n$ spans range T.

Since we found a list that spans range T, we conclude that range T is finite-dimensional

Now we will show that $Tv_1, \dots, Tv_n$ is linearly independent

Suppose $c_1, \dots, c_n \in \mathbb{F}$ that satisfy

$$c_1 Tv_1 + \cdots + c_n Tv_n = 0$$

Then we have

$$0 = c_1 Tv_1 + \cdots + c_n Tv_n$$

$$= T(c_1 v_1) + \cdots + T(c_n v_n)$$

$$= T(c_1 v_1 + \cdots c_n v_n)$$

Therefore, $c_1 v_1 + \cdots c_n v_n \in$ null T

Since $u_1, \dots, u_m$ is a basis of null T, it spans null T, so we can write

vector in null T

$$\boxed{c_1 v_1 + \cdots + c_n v_n} = d_1 u_1 + \cdots + d_m u_m$$

for some $d_1, \dots d_m \in \mathbb{F}$. So we get

$$-d_1 v_1 - \cdots - d_m u_m + c_1 v_1 + \cdots + c_n v_n = 0$$

But $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis of V so it is linearly independent in V. So all scalars are zero

$$-d_1 = u_1 \dots, -d_m = 0, c_1 = 0, \dots, c_n = 0$$

In particular,

$$c_1 = 0, \dots, c_n = 0$$

So $Tv_1, \dots, Tv_n$ is linearly independent

So it is a basis of range of T. So $\dim(\text{range } T) = n$

In summary, we have

$$\dim V = m + n$$
$$\dim(\operatorname{null} T) = m$$
$$\dim(\operatorname{range} T) = n$$

Therefore

$$\begin{aligned}\dim V &= m+n \\ &= \dim(\operatorname{null} T) + \dim(\operatorname{range} T)\end{aligned}$$

as desired.

3.23 A map to a smaller dimensional space is not injective

Suppose V and W are finite-dimensional vector spaces that satisfy $\dim V > \dim W$. Then no linear map from V to W is injective.

Proof: suppose that we have $T \in \mathcal{L}(V, W)$

Then, by the fundamental theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim(\operatorname{null} T) &= \dim V - \dim(\operatorname{range} T) \\ &\geq \dim V - \dim W \\ &> \dim W - \dim W \\ &= 0\end{aligned}$$

range T is a subspace of W by 2.38 of Axler $\Rightarrow \dim(\operatorname{range} T) \leq \dim W$

So $\operatorname{null} T \neq \{0\}$. By 3.16 of Axler, T is not injective

3.24 A map to a larger dimensional space is not surjective

Suppose V and W are finite-dimensional vector spaces that satisfy $\dim V < \dim W$. Then no linear map from V to W is surjective.

Proof: Let $T \in \mathcal{L}(V, W)$. Then, by the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim(\operatorname{range} T) &= \dim V - \dim \operatorname{null} T \\ &\leq \dim V - 0 \\ &= \dim V \\ &< \dim W\end{aligned}$$

$0 \in \operatorname{null} T$
by 2.38 Axler $\Rightarrow \{0\} \subset \operatorname{null} T$
$\dim \{0\} \leq \dim \operatorname{null} T$
$0 \leq \dim \operatorname{null} T$
$(\dim \operatorname{null} T \geq 0)$

By Exercise 2.C1 of Axler, range $T \neq W$.

Therefore T is not surjective

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3.25 Example

fix positive integers m, n. Consider the homogeneous system of linear equations

a system of m equations
$$\begin{cases} \sum_{k=1}^{n} A_{1,k} x_k = 0 \\ \vdots \\ \sum_{k=1}^{n} A_{m,k} x_k = 0 \end{cases}$$

j^{th} equation

$$\sum_{k=1}^{n} A_{j,k} x_k = 0$$

for some $A_{j,k} \in \mathbb{F}$, for $j = 1, \ldots, m$ and for $k = 1, \ldots, n$

Rephrase in terms of a linear map the equation of whether this system of equations has a nonzero solution

Proof: Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$\underbrace{T(x_1, \ldots, x_n)}_{n \text{ coordinates}} = \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots \sum_{k=1}^{n} A_{m,k} x_k \right)}_{m \text{ coordinates}}$$

Then the equation $\left(\sum_{k=1}^{n} A_{1,k} x_k, \sum_{k=1}^{n} A_{m,k} x_k \right)$

$$T\underbrace{(0, \ldots, 0)}_{n} = \underbrace{(0, \ldots, 0)}_{m}$$

is equivalent to the homogeneous system of equations

Note that

$(x_1, \ldots, x_n) = (0, \ldots, 0)$ is a solution to the homogeneous system of equations. This is equivalent to saying

$$T\underbrace{(0, \ldots, 0)}_{n} = \underbrace{(0, \ldots, 0)}_{m}$$

$x_1 = 0$

$\vdots$

$x_n = 0$

There exist nonzero solutions $x_1, \ldots, x_n$ to the homogeneous system of equations if and only if

$$\text{null } T \neq \{0\}$$

In other words, if and only if there exist $x_1, \ldots, x_n$, not all zero such that $(x_1, \ldots, x_n) \in \text{null } T$.

3.26 Homogeneous system of linear equations

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Example

$$\sum_{k=1}^{n} A_{1,k}x_k = 0$$

$$\sum_{k=1}^{n} A_{2,k}x_k = 0$$

And x_1, x_2, x_3 solve the system of equations, then at least one of x_1, x_2, x_3 is nonzero.

Proof: Again define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$\underbrace{T(x_1, \dots, x_n)}_{n} = \underbrace{\left(\sum_{k=1}^{n} A_{1,k}x_k, \dots, \sum_{k=1}^{n} A_{m,k}x_k\right)}_{m}$$

Then we have a homogeneous system of m linear equations with n variables $x_1, \dots, x_n$. Since there are more variables then there are equations -

we have $\dim \mathbb{F}^n = n > m = \dim \mathbb{F}^m$

By 3.23 of Axler, $T: \mathbb{F}^n \to \mathbb{F}^m$ is not injective

By 3.16 of Axler, null $T \neq \underbrace{\{(0, \dots, 0)\}}$. So there exist nonzero solutions of the system of equations

3.27 Example

Rephrase in terms of a linear map the question of whether the in-homogeneous system of linear equations

$$\sum_{k=1}^{n} A_{1,k}x_k = c_1$$

$$\vdots$$

$$\sum_{k=1}^{n} A_{m,k}x_k = c_m$$

for any $A_{j,k} \in \mathbb{F}$ ($j = 1, \dots, m$ and $k = 1, \dots, m$) and for som $c_1, \dots, c_m \in \mathbb{F}$, has no solution

Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$\underbrace{T(x_1, \dots, x_n)}_{n \text{ coordinates}} \quad \underbrace{\left(\sum_{k=1}^{n} A_{1,k}x_k, \dots, \sum_{k=1}^{n} A_{m,k}x_k\right)}_{m \text{ coordinates}}$$

Then the equation

$$T(x_1, \dots, x_n) = (c_1, \dots, c_m)$$

is equivalent to the inhomogeneous system of equations

3.29 Inhomogeneous system of linear equations

An inhomogeneous system of linear equations with more equations than variables has no solution $\boxed{\text{for some}}$ choice of constent terms.

proof: Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$\underbrace{T(x_1, \dots, x_n)}_{n} = \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{m,k} x_k \right)}_{m}$$

Since there are more equations than there are variables we have

$$\dim \mathbb{F}^n = n < m = \dim \mathbb{F}^m$$

By 3.24 of Axler, $T: \mathbb{F}^n \to \mathbb{F}^m$ is NOT surjective.

In other words,

$$\text{range } T \neq \mathbb{F}^m$$

So there exist $c_1, \dots, c_m \in \mathbb{F}^m \setminus \text{range } T$ such that

$$T(x_1, \dots, x_n) \neq (c_1, \dots, c_m)$$

So we have

$$\sum_{k=1}^{n} A_{1,k} x_k \neq c_1$$

$$\sum_{k=1}^{n} A_{m,k} x_k \neq c_1$$

which means the system of inhomogeneous equations with this choice of $c_1, \dots, c_m$ does NOT contain any solutions.