3.B Null spaces and Ranges

3.12 Definition

If we have $T \in \mathcal{L}(V, W)$, then the <u>null space</u> of T is the subset of V consisting of vectors in V that T maps to 0.

$$\text{null } T = \{v \in V : Tv = 0\}$$

3.13 Examples

- Consider the zero map $0 \in \mathcal{L}(V, W)$. For all $v \in V$, we have

$$0v = 0.$$

Therefore, null $0 = \{v \in V : 0v = 0\} = 0$

$\rightarrow$ has to be $\mathbb{C}$

- Define $\varphi \in \mathcal{L}(\mathbb{C}^3, \mathbb{C})$ by $\varphi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$.

Then we have null $\varphi = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \varphi(z_1, z_2, z_3) = 0\}$.

$= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + 2z_2 + 3z_3 = 0\}$.

The basis of null φ is $(-2, 1, 0), (-3, 0, 1)$.

$$\begin{aligned}(z_1, z_2, z_3) &= (-2z_2, -3z_3, z_2, z_3) \\ &= (-2z_2, z_2, 0) + (-3z_3, 0, z_3) \\ &= z_2(-2, 1, 0) + z_3(-3, 0, 1)\end{aligned}$$

So $(-2, 1, 0), (-3, 0, 1)$ spans null φ.

Then prove that $(-2, 1, 0), (-3, 0, 1)$ is linearly independent.

So $(-2, 1, 0), (-3, 0, 1)$ is a basis of null φ.

- Let $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ be the differentiation map defined by $Dp = p'$.

Then we have null $D = \{p \in \mathcal{P}(\mathbb{R}) : p'(z) = 0 \text{ for all } z \in \mathbb{F}\}$.

$= \{p \in \mathcal{P}(\mathbb{R}) : p'(z) = c \text{ for all } z \in \mathbb{F}, \text{ for some constant } c\}$.

$= \{p \in \mathcal{P}(\mathbb{R}) : p \text{ is a constant function}\}$.

- Define $T \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ by $(Tp)(x) = x^2 p(x)$ for all $x \in \mathbb{R}$.

Then we have
$$\begin{aligned} \text{null } T &= \{p \in P(\mathbb{R}) : Tp = 0\} \\ &= \{p \in P(\mathbb{R}) : (Tp)(x) = 0 \text{ for all } x \in \mathbb{R}\}. \\ &= \{p \in P(\mathbb{R}) : x^2 p(x) = 0 \text{ for all } x \in \mathbb{R}\}. \\ &= \{p \in P(\mathbb{R}) : p(x) = 0 \text{ for all } x \in \mathbb{R}\}. \\ &= \{p \in P(\mathbb{R}) : p = 0\}. \\ &= \{0\} \end{aligned}$$

The only polynomial such that $x^2 p(x) = 0$ for all $x \in \mathbb{R}$ is $p = 0$.

3.14 Null space is a subspace

Suppose we have $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

Proof:
- Additive identity

  Since we have $T \in \mathcal{L}(V, W)$, it follows that $T: V \to W$ is a linear map. By 3.11 of Axler, we have $T(0) = 0$.

  Therefore, $0 \in$ null T.

- Closed under addition

  Suppose we have $u, v \in$ null T. Then we have $Tu = 0$ and $Tv = 0$

  So we have $T(u+v) = Tu + Tv = 0 + 0 = 0$

  Therefore, $u+v \in$ null T.

- Closed under scalar multiplication.

  Suppose we have $u \in$ null T and $\lambda \in \mathbb{F}$. Then $Tu = 0$.

  So we have $T(\lambda u) = \lambda Tu = \lambda \cdot 0 = 0$.

  Therefore, $\lambda u \in$ null T.

So we conclude that null T is a subspace of V.

3.15 Definition

A function $T: V \to W$ is called <u>injective</u> if $Tu = Tv$ implies $u = v$, for all $u, v \in V$.

↳ also called one-one

3.16 Injectivity is equivalent to null space equals $\{0\}$.

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$.

Proof: • <u>Forward direction: If T is injective, then null $T = \{0\}$.</u>

Suppose T is injective. Since T is also linear, we have $T(0) = 0$, which means $0 \in$ null T, and so $\{0\} \subset$ null T.

We will prove null $T \subset \{0\}$. Suppose $v \in$ null T.

Then $Tv = 0$. But we also have $T(0) = 0$.

Therefore, $Tv = T(0)$. ($Tv = 0 = T(0)$).

Since T is injective, $v = 0$. In other words, $v \in \{0\}$.

So null $T \subset \{0\}$. So we get the set equality null $T = \{0\}$.

• <u>Backward direction: If null $T = \{0\}$, then T is injective</u>

Suppose null $T = \{0\}$. Suppose $u, v \in V$ satisfy $Tu = Tv$. Then we have

$$\begin{aligned} 0 &= Tu - Tv \\ &= T(u - v) \end{aligned}$$

Therefore, $u - v \in$ null T. But null $T = \{0\}$. So $u - v \in \{0\}$, which implies $u - v = 0$, or equivalently, $u = v$. So T is injective.

Range and subjectivity

3.17 Definition

The <u>range</u> of a function $T: V \to W$ is a subset of W consisting of all vectors of the form Tv for some $v \in V$ and is denoted

$$\text{range } T = \{Tv : v \in V\}.$$

3.18 Example

- Consider the zero map $0: V \longrightarrow W$. Then $0u = 0$ for all $u \in V$.

  So we have $\text{range } 0 = \{0u : u \in V\}$.
  $= \{0 : u \in V\}$.
  $= \{0\}$.

- Define $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ by $T(x,y) = (2x, 5y, x+y)$.

  Then we have $\text{range } T = \{T(x,y) : (x,y) \in \mathbb{R}^2\}$
  $= \{(2x, 5y, x+y) : (x,y) \in \mathbb{R}^2\}$.

  A basis of range T is $(2,0,1), (0,5,1)$

  $T(x,y) = (2x, 5y, x+y) = (2x, 0, x) + (0, 5y, y) = x(2,0,1) + y(0,5,1)$

  So $(2,0,1), (0,5,1)$ spans range T.

  Show that $(2,0,1), (0,5,1)$ is also linearly independent.

  Then $(2,0,1), (0,5,1)$ is a basis of range T.

- Let $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$, be the differentiation map defined by $Dp = p'$.

  For each polynomial $q \in P(\mathbb{R})$, there exists a polynomial $p \in P(\mathbb{R})$ that satisfies $p' = q$.

  So we have $\text{range } D = \{Dp : p \in P(\mathbb{R})\}$.
  $= \{p' : p \in P(\mathbb{R})\}$.
  $= \{q : q \in P(\mathbb{R})\}$.
  $= P(\mathbb{R})$.

3.19 The range is a subspace

If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W.

Proof: - Additive identity

Suppose we have $T \in \mathcal{L}(V, W)$. Then, by 3.11 of Axler.
we have $T(0) = 0$. So $0 \in \text{range } T$.

- Closed under addition

Suppose $w_1, w_2 \in$ range T. Then $w_1 = Tv_1$ and $w_2 = Tv_2$ for some $v_1, v_2 \in V$.

So we have $T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2$.

Since $v_1 + v_2 \in V$, it follows that we have $w_1 + w_2 \in$ range T.

- Closed under multiplication

Suppose $\lambda \in \mathbb{F}$ and $w \in$ range T. Then $w = Tv$ for some $v \in V$.

So we have $T(\lambda v) = \lambda Tv = \lambda w$.

Since $\lambda v \in V$, it follows that we have $\lambda w \in$ range T.

So range T is a subspace of W.

3.22 Fundamental Theorem of Linear Maps

Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and $\dim V = \dim \text{null } T + \dim \text{range } T$.

3.20 Definition

A function $T: V \to W$ is a surjective if its range equals W; that is,

$$\text{range} T = W.$$

Proof: Let ~~u_1~~ $u_1, \ldots, u_m$ be a basis of null T. Then $\dim(\text{null} T) = m$.

Also, $u_1, \ldots, u_m$ is a linearly independent list. By 2.33 of Axler, we can extend this list to a basis

$$u_1, \ldots, u_m, v_1, \ldots, v_n.$$

of V. Then $\dim V = m + n$.

So we need to just show that ~~range~~ range T is finite-dimensional, with $\dim(\text{range } T) = n$. To accomplish this goal, we need to show that $Tv_1, \ldots, Tv_n$ is a basis of range T.

Let $v \in V$ arbitrary. Since $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V, it spans V.

So we can write $v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n$.

for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$. So we have

$$
\begin{aligned}
& Tv = T(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) \\
\text{(additivity of } T) \to \; & = T(a_1u_1) + \cdots + T(a_mu_m) + T(b_1v_1) + \cdots + T(b_nv_n) \\
\text{(homogeneity of } T) \to \; & = a_1Tu_1 + \cdots + a_mTu_m + b_1Tv_1 + \cdots + b_nTv_n \\
\left(\begin{array}{c} u_1, \ldots, u_m \in \operatorname{null} T \\ (u_1, \ldots, u_m \text{ is a basis} \\ \text{of null } T) \end{array}\right) \to \; & = a_1 0 + \cdots + a_m 0 + b_1Tv_1 + \cdots + b_nTv_n \\
& = b_1Tv_1 + \cdots + b_nTv_n
\end{aligned}
$$

Therefore, the list $Tv_1, \ldots, Tv_n$ spans range T. Since we found a list that spans range T, we conclude that range T is finite-dimensional.

Now we will show that $Tv_1, \ldots, Tv_n$ is linearly independent.

Suppose $c_1, \ldots, c_n \in \mathbb{F}$ that satisfy $c_1Tv_1 + \cdots + c_nTv_n = 0$.

Then we have

$$
\begin{aligned}
0 &= c_1Tv_1 + \cdots + c_nTv_n \\
&= T(c_1v_1) + \cdots + T(c_nv_n) \\
&= T(c_1v_1 + \cdots + c_nv_n)
\end{aligned}
$$

Therefore, $c_1v_1 + \cdots + c_n \in \operatorname{null} T$.

Since $u_1, \ldots, u_m$ is a basis of null T, it spans null T, so we can write

$$\overbrace{\boxed{c_1v_1 + \cdots + c_nv_n}}^{\text{vector in null } T} = d_1u_1 + \cdots + d_mu_m \quad \text{for some } d_1, \ldots, d_m \in \mathbb{F}.$$

So we get $\cancel{c_1v_1 + \cdots + c_nv_n}$ $-d_1u_1 + \cdots - d_mu_m + c_1v_1 + \cdots + c_nv_n = 0$.

But $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V.

So all scalars are $\cancel{\neq}$ zero: $-d_1 = 0, \ldots, -d_m = 0$, $c_1 = 0, \ldots, c_n = 0$

In particular, $c_1 = 0, \ldots, c_n = 0$.

So $Tv_1, \ldots, Tv_n$ is linearly independent.

So it is a basis of range T.

So $\dim(\operatorname{range} T) = n$.

In summary, we have

$$\dim V = m+n$$
$$\dim(\text{null } T) = m$$
$$\dim(\text{range } T) = n.$$

Therefore, $\dim V = m+n = \dim(\text{null } T) + \dim(\text{range } T)$, as desired.

3.23 A map to a smaller dimensional space is not injective

Suppose V and W are finite-dimensional vector spaces that satisfy $\dim V > \dim W$. Then no linear map from V to W is injective.

<u>Proof</u>: Suppose ~~by contradiction~~ that we have $T \in \mathcal{L}(V, W)$.

Then, by the Fund. Thm of Linear Maps (3.22 of Axler), we have

$$\begin{aligned} \dim(\text{null } T) &= \dim V - \dim(\text{range } T) \\ &\geq \dim V - \dim W \\ &> \dim W - \dim W \\ &= 0 \end{aligned}$$

T is a subspace of W
by 2.38 of Axler
$\Longrightarrow$ $\dim(\text{range } T) \leq \dim W$

So $\text{null } T \neq \{0\}$. By 3.16 of Axler, T is not injective.

3.24 A map to a larger dimensional space is not surjective

Suppose V and W are finite-dimensional vector spaces that satisfy $\dim V < \dim W$. Then no linear map from V to W is surjective

<u>Proof</u>: Let $T \in \mathcal{L}(V, W)$. Then, by the Fund. Thm of Linear Maps (3.22 of Axler), we have

$$\begin{aligned} \dim(\text{range } T) &= \dim V - \dim \text{null } T \\ &\leq \dim V - 0 \\ &= \dim V. \\ &< \dim W. \end{aligned}$$

$0 \in \text{null } T$
$\Rightarrow \{0\} \subset \text{null } T$
by 2.38 Axler $\Longrightarrow$ $\dim\{0\} \leq \dim(\text{null } T)$
$0 \leq \dim(\text{null } T)$
$(\dim(\text{null } T) \geq 0)$

By Exercise 2.C1 of Axler, $\text{range } T \neq W$.

Therefore, T is not surjective.

3.25 Example

Fix posetive integers m, n

Consider the homogenous system of linear equations

$$\text{a system of } m \text{ equations} \begin{cases} \sum_{k=1}^{n} A_{1,k} x_k = 0 \\ \vdots \\ \sum_{k=1}^{n} A_{m,k} x_k = 0 \end{cases}$$

jth equation

$$\sum_{k=1}^{n} A_{j,k} x_k = 0$$

for some $A_{j,k} \in \mathbb{F}$, for $j = 1, \ldots, m$ and for $k = 1, \ldots, n$.

Rephase in therms of a linear map the question of whether this system of equations has a nonzero solution.

Proof: Define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$\underbrace{T(x_1, \ldots, x_n)}_{n \text{ coordinates}} = \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k \right)}_{m \text{ coordinates}}$$

The equation: $\underbrace{T(0, \ldots, 0)}_{n} = \underbrace{(\overset{\sum_{k=1}^{n} A_{1,k} x_k}{\overset{\shortparallel}{0}}, \ldots, \overset{\sum_{k=1}^{n} A_{m,k} x_k}{\overset{\shortparallel}{0}})}_{m}$

is equivalent to the homogeneous system of equators.

Note that $(x_1, \ldots, x_n) = (0, \ldots, 0)$ is a solution to the homogeneous system of equations. This is equivalent to system

$$\underbrace{T(0, \ldots, 0)}_{n} = \underbrace{(0, \ldots, 0)}_{m}$$

$$\begin{matrix} x_1 = 0 \\ \vdots \\ x_n = 0 \end{matrix}$$

There exist nonzero solutions $x_1, \ldots, x_n$ to the homogeneous system of equations if and only if $\text{null}\, T \neq \{0\}$.

In other words, if and only if there exist $x_1, \ldots, x_n$, not all zero such that $(x_1, \ldots, x_n) \in \text{null}\, T$.

3.26 Homogeneous system of linear equations

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Proof: Again, define $T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$\underbrace{T(x_1, \ldots, x_n)}_{n} = \underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k \right)}_{m}$$

Then we have a homogeneous system of m linear equations with n variables $x_1, \ldots, x_n$. Since there are more variables than there are equations, we have

$$\dim \mathbb{F}^n = n > m = \dim \mathbb{F}^m.$$

By 3.23 of Axler, $T: \mathbb{F}^n \to \mathbb{F}^m$ is not injective.

By 3.16 of Axler, $\operatorname{null} T \neq \{\underbrace{(0, \ldots, 0)}_{n}\}$.

So there exist nonzero solutions of the system of equations.

Example: $\sum_{k=1}^{n} A_{1,k} x_k = 0$

$\sum_{k=1}^{n} A_{2,k} x_k = 0$

And x_1, x_2, x_3 solve the above systems of equations, then at least one of x_1, x_2, x_3 is nonzero.

3.27 Example

Rephase in terms of a linear map the question of whether the in homogeneous system of linear equations

$$\sum_{k=1}^{n} A_{1,k} x_k = c_1$$

$$\vdots$$

$$\sum_{k=1}^{n} A_{m,k} x_k = c_m,$$

for any $A_{j,k} \in \mathbb{F}$ ($j=1, \ldots, m$ and $k=1, \ldots, m$) and for some $c_1, \ldots, c_m \in \mathbb{F}$, has no solution.

Define $T: \mathbb{F}^n \longrightarrow \mathbb{F}^m$ by $T\underbrace{(x_1, \ldots, x_n)}_{n \text{ coordinates}}$ $\underbrace{\left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k\right)}_{m \text{ coordinates.}}$

Then the equation $T(x_1, \ldots, x_n) = (c_1, \ldots, c_m)$
is equivalent to the inhomogeneous system of equations.

3.29 Inhomogeneous system of linear equations

An inhomogeneous system of linear equations with more equations than the variables has no solution $\boxed{\text{for some}}$ choice of constant terms

Proof: Define $T: \mathbb{F}^n \longrightarrow \mathbb{F}^m$ by

$$T\underbrace{(x_1, \ldots, x_n)}_{n} = \underbrace{\Big(\overset{c_1}{\overset{\shortparallel}{\sum_{k=1}^{n} A_{1,k} x_k}}, \ldots, \overset{c_m}{\overset{\shortparallel}{\sum_{k=1}^{n} A_{m,k} x_k}}\Big)}_{m}$$

Since there are more equations than there are variables, we have

$$\dim \mathbb{F}^n = n < m = \dim \mathbb{F}^m.$$

By 3.24 of Axler, $T: \mathbb{F}^n \longrightarrow \mathbb{F}^m$ is NOT surjective. In other words, range $T \neq \mathbb{F}^m$. So there exist $c_1, \cdots, c_m \in \mathbb{F}^m \setminus \text{range } T$ such that

$$T(x_1, \ldots, x_n) \neq (c_1, \ldots, c_m).$$

So we have
$$\sum_{k=1}^{n} A_{1,k} x_k \neq c_1$$
$$\vdots$$
$$\sum_{k=1}^{n} A_{m,k} x_k \neq c_m,$$
Which means the system of inhomogeneous equations with this choice of $c_1, \ldots, c_m$ does NOT contain any solutions.