3.C Matrices

3.30 Def

Let m & n denote positive ints.

An $m \times n$ matrix A is a rectangular array of elements of $\mathbb{F}$, with m rows & n columns.

3.31 Example

If $A = \begin{pmatrix} 8 & 4 & 5-3i \\ 1 & 9 & 7 \end{pmatrix}$

then $A_{1,1} = 8$ $A_{1,3} = 5-3i$ $A_{2,3} = 7$

3.32 Def

Suppose $T \in \mathcal{L}(V, W)$ & $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The matrix of T with respect to these bases is the $m \times n$ matrix:

$$M(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k} w_1 + \ldots + A_{m,k} w_m$$

If the bases are understood, we can write $M(T)$ to denote the matrix T with respect to the bases.

For any $k = 1, \ldots, n$, we have Tv_k

$$M(T) = \begin{bmatrix} A_{1,1} & A_{1,k} & A_{1,n} \\ A_{j,1} & A_{j,k} & A_{j,n} \\ A_{m,1} & A_{m,k} & A_{m,n} \end{bmatrix} \qquad Tv_k = [w_1, \ldots, w_m] \begin{bmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{bmatrix}$$

$$= [A_{1,k} w_1 + \ldots + A_{m,k} w_m]$$

$$= A_{1,k} w_1 + \ldots + A_{m,k} w_m$$

$$= \sum_{j=1}^{m} A_{j,k} w_j$$

3.33 Example Define $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ by

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9x)$$

Find the matrix of T with respect to the standard bases of $\mathbb{F}^2, \mathbb{F}^3$.

In other words, find

$$M(T) = M(T, ((1,0), (0,1)), (1,0,0), (0,1,0), (0,0,1))$$

Solution: $T(1,0) = (1, 2, 7)$ $\qquad M(T) = \begin{vmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{vmatrix}$

$T(0,1) = (3, 5, 9)$

3.34 Example

let $1, x, x^2, \ldots, x^n$ be the standard basis of $P_m(\mathbb{F})$. Suppose $D \in \mathcal{L}(P_3(\mathbb{R}), P_2(\mathbb{R}))$ is the

differentiation map defined by $Dp = p'$
Find the matrix of D with respect to the standard bases of $P_3(\mathbb{R})$ & $P_2(\mathbb{R})$.
Find:

$$M(D) = M(D, \underbrace{(1, x, x^2, x^3)}_{\text{standard basis of } P_3(\mathbb{R})}, \underbrace{(1, x, x^2)}_{\text{standard basis of } P_2(\mathbb{R})})$$

Solution:
Basis of $P_3(\mathbb{R})$ is $1, x, x^2, x^3$
Basis of $P_2(\mathbb{R})$ is $1, x, x^2$
$\forall$ positive ints n, $D(x^n) = (x^n)' = n x^{n-1}$

Input of $P_3(\mathbb{R})$	Corresponding vector $\mathbb{F}^4$	Output	Corresponding vector $\mathbb{F}^3$
1	$(1, 0, 0, 0)$	$D(1) = 0$	$(0, 0, 0)$
x	$(0, 1, 0, 0)$	$D(x) = 1$	$(1, 0, 0)$
x^2	$(0, 0, 1, 0)$	$D(x^2) = 2x$	$(0, 2, 0)$
x^3	$(0, 0, 0, 1)$	$D(x^3) = 3x^2$	$(0, 0, 3)$

$$M(D) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix}$$

Input in $P_3(\mathbb{R})$		Output in $P_2(\mathbb{R})$
1	$M(D) \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$	0
x	$M(D) = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}$	1
x^2	$M(D) = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 2 \\ 0 \end{vmatrix} = 2 \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}$	$2x$
x^3	$M(D) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 3 \end{vmatrix} = 3 \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$	$3x^2$

Verify differentiation with map $D(1) = 0$ $D(x) = 1$ $D(x^2) = 2x$ $D(x^3) = 3x^2$

7/15 3.C Continued.

Addition & scalar multiplication of matrices

3.35 Def

Matrix Addition is the sum of 2 matrices with the same size, obtained by adding the corresponding entries in the matrices:

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} + \begin{bmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{bmatrix} = \begin{bmatrix} A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n} \end{bmatrix}$$

In other words, $A_{j,k} + C_{j,k} = (A+C)_{j,k}$ $\exists$ $j=1,\ldots,m$ and $k=1,\ldots,n$

3.36 The matrix of the sum of linear maps

Suppose $S,T \in \mathcal{L}(V,W)$. Then $M(S+T) = M(S) + M(T)$

3.37 Def

Scalar multiplication of a matrix - the product of a scalar and a matrix, obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} = \begin{bmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{bmatrix}$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$, $\exists$ $j=1,\ldots,m$ & $k=1,\ldots,n$

3.38 The matrix of a scalar times a linear map

Suppose $\lambda \in \mathbb{F}$ & $\mathcal{L}(V,W)$. Then $M(\lambda T) = \lambda M(T)$

3.40 $\dim \mathbb{F}^{m,n} = mn$

let m & n be positive ints. Let $\mathbb{F}^{m,n}$ is the set of all $m \times n$ matrices with entries in $\mathbb{F}$.

The dim $\mathbb{F}^{m,n} = mn$

* Proof: $\mathbb{F}^{m,n}$ is a vector space with respect to the operations of matrix addition & scalar multiplication of a matrix.

Now, dim $\mathbb{F}^{m,n} = mn$ because

$$
m\left\{
\underbrace{
\begin{array}{llll}
\begin{vmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{vmatrix}, &
\begin{vmatrix} 0 & \cdots 1 \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{vmatrix}, & \cdots, &
\begin{vmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{vmatrix}, \\
\begin{vmatrix} 0 & \cdots & 0 \\ 1 & 0 & \vdots \\ 0 & \cdots & 0 \end{vmatrix}, &
\begin{vmatrix} 0 & \cdots & 0 \\ 0 & 1 & \vdots \\ 0 & \cdots & 0 \end{vmatrix}, & \cdots, &
\begin{vmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & 1 \\ 0 & \cdots & 0 \end{vmatrix}, \\
\begin{vmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{vmatrix}, &
\begin{vmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots 1 \cdots & 0 \end{vmatrix}, & \cdots, &
\begin{vmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{vmatrix}
\end{array}
}_{n}
\right.
$$

The above is a basis of $\mathbb{F}^{m,n}$. So dim $\mathbb{F}^{m,n} = mn$

Matrix Multiplication

let $v_1, \dots, v_n$ be an basis of V & $w_1, \dots, w_m$ be a basis of W.

And let U be a vector space and $u_1, \dots, u_p$ is a basis of U.

let $T: U \to V$ & $S: V \to W$ be a linear maps.

We will show: $M(ST) = M(S)M(T)$

Suppose $M(S) = A$ & $M(T) = C$ $\exists$ $k = 1, \dots, p$ we have.

$$(ST)u_k = S(Tu_k) = S\left(\sum_{r=1}^{n} C_{r,k} v_r\right) \text{ by 3.32}$$

$$= \sum_{r=1}^{n} C_{r,k} S v_r \text{ since } S \text{ is linear.}$$

$$= \sum_{r=1}^{n} C_{r,k}\left(\sum_{j=1}^{m} A_{j,r} w_j\right) \text{ by 3.32}$$

$$= \sum_{j=1}^{m}\left(\sum_{r=1}^{n} A_{j,r} C_{r,k}\right) w_j$$

By 3.32, $D = M(ST)$ is the $m \times p$ matrix with entry in row j, column k, $D_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$

3.41 [Def]

Let A be an $m \times n$ matrix & C be an $n \times p$ matrix. Then AC is an $m \times p$ matrix with entry in row j, column k: $(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$

In other words,

$$AC = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} \begin{bmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{r=1}^{n} A_{1,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r} C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r} C_{r,p} \end{bmatrix}$$

3.42 Example

$$\begin{array}{ccccc} A & & C & & AC \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} & & \begin{bmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{bmatrix} & = & \begin{bmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{bmatrix} \\ 3\times 2 & & 2 \times 4 & & 3 \times 4 \end{array}$$

3.43 The matrix of the product of linear maps

If $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$, then

$$M(ST) = M(S)M(T)$$

Let A be an $m \times n$ matrix

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & \ddots & & \ddots & \vdots \\ A_{j,1} & & A_{j,k} & & A_{j,n} \\ \vdots & \ddots & & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{bmatrix}$$

Then $A_{j,\cdot} = (A_{j,1} \cdots A_{j,n})$

☆ $\cdot$ = row $\quad A_{\cdot,k} = \begin{bmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{bmatrix}$

3.45 Example

Let $A = \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix}$. then $A_{2,\cdot} = \begin{bmatrix} 1 & 9 & 7 \end{bmatrix}$ & $A_{\cdot,2} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

3.46 Example

Let $A = \begin{pmatrix} 3 & 4 \end{pmatrix}$ & $C = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ Then

$AC = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 26 \end{pmatrix} = 26$

$(AC)_{1,1} = 26 \quad (AC)_{1,\cdot} = 26 \quad (AC)_{\cdot,1} = 26$

3.47 Entry of matrix product equals row times column

Suppose A is a $m \times n$ matrix and C is an $n \times p$ matrix. Then $(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k}$

Proof: $(AC)_{j,k} = \left(\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C & \cdots & C_{n,p} \end{pmatrix} \right)_{j,k}$

$= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r} C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r} C_{r,p} \end{pmatrix}_{j,k}$

$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$

$A_{j,\cdot} C_{\cdot,k} = (A_{j,1} \cdots A_{j,n}) \begin{pmatrix} C_{1,k} \\ C_{n,k} \end{pmatrix}$

$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$

So, $(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k}$ ∎

3.50 Example

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}} \underset{C_{\cdot,2}}{\begin{bmatrix} 5 \\ 1 \end{bmatrix}} = \underset{(AC)_{\cdot,2}}{\begin{bmatrix} 7 \\ 19 \\ 31 \end{bmatrix}}$$

3.52 linear Combination of Columns

Suppose A is an $m \times n$ and $C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is an $n \times 1$ matrix

Then $AC = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$

Proof: $c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n} = c_1 \begin{bmatrix} A_{1,1} \\ \vdots \\ A_{m,1} \end{bmatrix} + \cdots + \begin{bmatrix} A_{1,n} \\ \vdots \\ A_{m,n} \end{bmatrix}$

$$= \begin{bmatrix} c_1 A_{1,1} \\ \vdots \\ c_1 A_{m,1} \end{bmatrix} + \begin{bmatrix} c_n A_{1,n} \\ \vdots \\ c_n A_{m,n} \end{bmatrix}$$

$$= \begin{bmatrix} c_1 A_{1,1} + \cdots + c_n A_{1,n} \\ \vdots \\ c_1 A_{m,1} + \cdots + c_n A_{m,n} \end{bmatrix}$$

$$AC = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \begin{bmatrix} A_{1,1} c_1 + \cdots + A_{1,n} c_n \\ \vdots \\ A_{m,1} c_1 + \cdots + A_{m,n} c_n \end{bmatrix}$$

So, $AC = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$ $\blacksquare$

Similar exercise 3.C #11.