July. 15 2019.

3.C.

Addition and scalar multiplication of matrices.

3.35 Definition.

Matrix addition is the sum of two matrices of the same size, obtained adding the corresponding entries in the matrices.

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} + \begin{bmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{bmatrix} = \begin{bmatrix} A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n} \end{bmatrix}$$

In other words, $A_{j,k}+C_{j,k}=(A+C)_{j,k}$ for each $j=1,\dots,m$, $k=1,\dots,n$

3.36 The matrix of the sum of linear maps.

Suppose $S,T\in\mathcal{L}(V,W)$. Then $\mathcal{M}(S+T)=\mathcal{M}(S)+\mathcal{M}(T)$

3.37 Definition

Scalar multiplication of a matrix is the product of a scalar and a matrix, obtained by multiplying each entry in the matrix by the scalar:

$$\lambda\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} = \begin{bmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{bmatrix}$$

In other words, $(\lambda A)_{j,k}=\lambda A_{j,k}$ for each $j=1,\dots,m$; $k=1,\dots,n$.

3.38 The matrix of a scalar times a linear map.

Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$. Then $M(\lambda T) = \lambda M(T)$

3.40 $\dim \mathbb{F}^{m,n} = mn$

Let m and n be positive integers. let $\mathbb{F}^{m,n}$ is the set of all $m \times n$ matrices with entries in $\mathbb{F}$. Then $\dim \mathbb{F}^{m,n} = mn$.

Proof:

$\mathbb{F}^{m,n}$ is a vector space with respect to operations of matrix addition and scalar multiplication of a matrix.

Now $\dim \mathbb{F}^{m,n} = mn$ because

$$\underbrace{\begin{bmatrix} \overbrace{\boxed{1 \cdots 0}}^{n} \\ \vdots \ \ddots \ \vdots \\ 0 \cdots 0 \end{bmatrix}, \begin{bmatrix} 0, 1 \cdots 0 \\ \vdots \ \ddots \ \vdots \\ 0 \cdots 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \cdots 1 \\ \vdots \ \ddots \ \vdots \\ 0 \cdots 0 \end{bmatrix}}_{n \text{ matrices}},$$

m-rows

n-columns

m matrices

$$\begin{bmatrix} 0, \cdots, 0 \\ 1, 0 \cdots 0 \\ 0, \cdots, 0 \end{bmatrix}, \begin{bmatrix} 0 \cdots 0 \\ 0\ 1 \cdots 0 \\ \vdots \ \ddots \ \vdots \\ 0 \cdots 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \cdots 0 \\ \vdots \ \ddots \ 1 \\ 0 \cdots 0 \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} 0 \cdots 0 \\ \vdots \ \ddots \ \vdots \\ 1 \cdots 0 \end{bmatrix}, \begin{bmatrix} 0 \cdots 0 \\ \vdots \ \ddots \ \vdots \\ 0\ 1 \cdots 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 \cdots 0 \\ \vdots \ \ddots \ \vdots \\ 0 \cdots 1 \end{bmatrix}$$

is a basis of $\mathbb{F}^{m,n}$ so $\dim \mathbb{F}^{m,n} = m \cdot n$.

Matrix multiplication

Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_m$ be a basis of W.

Also let U be a vector space and $u_1, \ldots, u_p$ is a basis of U.

Let $T: U \to V$ and $S: V \to W$ be linear maps.

We will show : $M(ST) = M(S)M(T)$.

Suppose $M(S) = A$ and $M(T) = C$. For each $k = 1, \dots, p$ we have

$$(ST)u_k = S\left(\sum_{r=1}^{n} C_{r,k} v_r\right)$$ by Definition 3.32 Axler

$$= \sum_{r=1}^{n} C_{r,k} v_r$$ Since S is linear.

$$= \sum_{r=1}^{n} C_{r,k}\left(\sum_{r=1}^{n} A_{j,k} w_j\right)$$

$$= \sum_{r=1}^{m}\left(\sum_{r=1}^{n} A_{j,r} C_{r,k}\right) w_j$$

By definition 3.32 of Axler, $M(ST)$ is the $m \times p$ matrix with entry in row j, column k.

$$D_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

3.41 Definition

Let A be an $m \times n$ matrix and C be an $n \times p$ matrix.

Then AC is an $m \times p$ matrix with entry row j, column k.

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

In other words,

$$AC = \begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix} \cdot \begin{bmatrix} C_{1,1} & \dots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \dots & C_{n,p} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{r=1}^{n} A_{1,r} C_{r,1} & \dots & \sum_{r=1}^{n} A_{1,r} C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r} C_{r,1} & \dots & \sum_{r=1}^{n} A_{m,r} C_{r,p} \end{bmatrix}$$

3.42 Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \end{bmatrix}$$

$\begin{bmatrix} 5 & 6 \end{bmatrix}$ $\quad 2\times4 \quad$ $\begin{bmatrix} 42 & 31 & 20 & 9 \end{bmatrix}$

$3\times2 \qquad\qquad\qquad 3\times4$

3.43 The matrix of the product of linear maps.

If $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$ then

$\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$

Let A be an $m\times n$ matrix.

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{bmatrix}$$

Then $A_{j,\cdot} = (A_{j,1} \cdots A_{j,n})$

$$A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

3.45 Example

Let $A = \begin{bmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{bmatrix}$ Then $A_{2,\cdot} = \begin{bmatrix} 1 & 9 & 7 \end{bmatrix}$

$A_{\cdot,2} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

3.46 Example

Let $A = (3 \;\; 4)$ and $C = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

$AC = (3\;4)\begin{pmatrix} 6 \\ 2 \end{pmatrix} = (26) = 26$

$(AC)_{1,1} = 26$, $(AC)_{1,\cdot} = 26$, $(AC)_{\cdot,1} = 26$

3.47 Entry of matrix product equals row times column.

Suppose A is an $m\times n$ matrix and C is an $n\times p$ matrix.

Then $(AC)_{j,k} = A_{j,\cdot} \cdot C_{\cdot,k}$

Proof : $(AC)_{j,k} = \left(\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} \begin{bmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{bmatrix} \right)_{j,k}$

$$= \begin{vmatrix} \sum_{r=1}^{n} A_{1,r}C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r}C_{r,p} \\ \vdots & & \vdots \\ \sum_{r=1}^{n} A_{m,r}C_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r}C_{r,p} \end{vmatrix}_{j,k}$$

$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

$$A_{j,\cdot} \cdot C_{\cdot,k} = (A_{j,1} \cdots A_{j,n}) \begin{pmatrix} C_{1,k} \\ \vdots \\ C_{n,k} \end{pmatrix}$$

$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

$\therefore (AC)_{j,k} = A_{j,\cdot} \cdot C_{\cdot,k}$

3.50 Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \\ 31 \end{bmatrix}$$

3.41 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{pmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix}$: $AC = \begin{bmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{bmatrix}$

3.52 Linear combination of columns.

Suppose A is an $m \times n$ matrix and $C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is an $n \times 1$ matrix.

Then $AC = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$.

Proof:

$$c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n} = c_1 \begin{pmatrix} A_{1,1} \\ \vdots \\ A_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} A_{1,n} \\ \vdots \\ A_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 A_{1,1} \\ \vdots \\ c_1 A_{m,1} \end{pmatrix} + \cdots + \begin{pmatrix} c_n A_{1,n} \\ \vdots \\ c_n A_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 A_{1,1} + \cdots + c_n A_{1,n} \\ \vdots \\ c_1 A_{m,1} + \cdots + c_n A_{m,n} \end{pmatrix}$$

$$AC = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1} c_1 + \cdots + A_{1,n} c_n \\ \vdots \\ A_{m,1} c_1 + \cdots + A_{m,n} c_n \end{pmatrix}$$

$$\therefore\ AC = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$$

3.D Invertibility and Isomorphic Vector Spaces.

Invertible linear maps.

3.53 Definition

- A linear map $T \in \mathcal{L}(V, W)$ is called invertible if there exists a linear map $S \in \mathcal{L}(W, V)$ s.t. $ST = I_V$ and $TS = I_W$, where I_V and I_W are identity maps on V and W respective.

identitu maps:

$I_V : V \Rightarrow V$ V, W vector spaces

$I_V(v) = v$ v, w vectors in V, W

$I_W : W \Rightarrow W$

$I_W(w) = w$

$T : V \Rightarrow W$

$S : W \Rightarrow V$

$\therefore ST : V \Rightarrow V$

$I_V : V \Rightarrow V$

If T is invertible with inverse S, then $ST = I_V$ and $TS = I_W$.

$TS = I_W$

$S : W \Rightarrow V$

$T : V \Rightarrow W$

$\therefore TS : W \Rightarrow W$

$(I_W : W \Rightarrow W)$

3.59 Inverse is unique

An invertible linear map has a unique inverse.

Proof:

Suppose $T \in \mathcal{L}(V, W)$ is invertible, and let S and $\tilde{S}$ be inverse of T.

Then $S = SI_W$ because $\tilde{S}$ is an inverse of T

$= S(T\tilde{S})$

$= (ST)\tilde{S}$ because S is an inverse of T.

$= I_V\tilde{S}$

$= \tilde{S}$

$\therefore S = \tilde{S}$ which means the inverse of T is unique.

If T is invertible, then its inverse is denoted by T^{-1}.
If $T \in \mathcal{L}(V,W)$ is invertible, then $T^{-1} = \mathcal{L}(W,V)$ is the unique element s.t.

$$T^{-1}T = I_V, \quad \text{and } TT^{-1} = I_W.$$