3.C Matrices

3.30 Definition:

Let m and n denote positive integers. An $m \times n$ matrix A is a rectangular array of elements of $\mathbb{F}$, with m rows and n columns:

$$A = \begin{array}{cl} \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix} & j\text{th row} \\ \quad\quad\quad\quad k\text{th column} & \end{array}$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A.

3.31 Example If $A = \begin{pmatrix} 8 & 4 & 5-3i \\ 1 & 9 & 7 \end{pmatrix}$,

then $A_{1,1} = 8$ $\quad A_{1,2} = 4$ $\quad A_{1,3} = 5-3i$

$A_{2,1} = 1$ $\quad A_{2,2} = 9$ $\quad A_{2,3} = 7$

3.32 Definition:

Suppose $T \in \mathcal{L}(V,W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The matrix of T with respect to these bases is the $m \times n$ matrix

$$\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the bases are understood, then we can just write $\mathcal{M}(T)$ denote the matrix of T with respect to the bases.

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ A_{m,1} & & A_{m,k} & & A_{m,n} \end{pmatrix}$$

for any $k = 1, \ldots, n$, we have

$$Tv_k = (w_1, \ldots, w_m) \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

$1 \times m$ $\qquad$ $m \times 1$

kth column of $\mathcal{M}(T)$

Identify all 1×1 matrices by their one entry

$$= (A_{1,k}w_1 + \cdots + A_{m,k}w_m)$$

1×1

$$= A_{1,k}w_1 + \cdots + A_{m,k}w_m$$

$$= \sum_{j=1}^{m} A_{j,k}w_j$$

3.33 Example | Define $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ by

$$T(x,y) = (x+3y,\ 2x+5y,\ 7x+9y).$$

Find the matrix of T with respect to the standard bases of $\mathbb{F}^2, \mathbb{F}^3$.

In other words, find

$$\mathcal{M}(T) = \mathcal{M}\Big(T, \underbrace{\boxed{((1,0),(0,1))}}_{\text{standard basis of } \mathbb{F}^2}, \underbrace{\boxed{((1,0,0),(0,1,0),(0,0,1))}}_{\text{standard basis of } \mathbb{F}^3}\Big)$$

with $v_1 = (1,0)$, $v_2 = (0,1)$, $w_1 = (1,0,0)$, $w_2 = (0,1,0)$, $w_3 = (0,0,1)$.

Solution: We have

v_1 $\quad T(1,0) = (1,2,7)$

v_2 $\quad T(0,1) = (3,5,9)$

The matrix of T with respect to the standard bases is

$$M(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} T(1,0), T(0,1) \end{pmatrix}$$

$$T(1,0) = Tv_1$$
$$= \sum_{j=1}^{3} A_{j,1} w_j$$
$$= A_{1,1} w_1 + A_{2,1} w_2 + A_{3,1} w_3$$
$$= 1(1,0,0) + 2(0,1,0) + 7(0,0,1)$$
$$= (1,2,7)$$

$$T(0,1) = Tv_2$$
$$= \sum_{j=1}^{3} A_{j,2} w_j$$
$$= A_{1,2} w_1 + A_{2,2} w_2 + A_{3,2} w_3$$
$$= 3(1,0,0) + 5(0,1,0) + 9(0,0,1)$$
$$= (3,5,9)$$

3.34 Example

Let $1, x, x^2, \ldots, x^m$ be the standard basis of $\mathcal{P}_m(\mathbb{F})$.
Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by

$$Dp = p'.$$

Find the matrix of D with respect to the standard bases of $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$. In other words, find

$$M(D) = M\Big(D, \underbrace{\boxed{1, x, x^2, x^3}}_{\text{standard basis of } \mathcal{P}_3(\mathbb{R})}, \underbrace{\boxed{1, x, x^2}}_{\text{standard basis of } \mathcal{P}_2(\mathbb{R})}\Big)$$

Solution: Basis of $P_3(\mathbb{R})$ is $1, x, x^2, x^3$.

Basis of $P_2(\mathbb{R})$ is $1, x, x^2$

For all positive integers n,

$$D(x^n) = (x^n)' = nx^{n-1}$$

Input in $P_3(\mathbb{R})$		corresponding vector
1	$\longleftrightarrow$	$(1,0,0,0)$
x	$\longleftrightarrow$	$(0,1,0,0)$
x^2	$\longleftrightarrow$	$(0,0,1,0)$
x^3	$\longleftrightarrow$	$(0,0,0,1)$

Output in $P_2(\mathbb{R})$		corresponding vector
$D(1) = 0$	$\longleftrightarrow$	$(0,0,0)$
$D(x) = 1$	$\longleftrightarrow$	$(1,0,0)$
$D(x^2) = 2x$	$\longleftrightarrow$	$(0,2,0)$
$D(x^3) = 3x^2$	$\longleftrightarrow$	$(0,0,3)$

$$M(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

input in $P_3(\mathbb{R})$		Output in $P_2(\mathbb{R})$
1	$M(D)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix}0\\0\\0\end{matrix}} & \begin{matrix}1\\0\\0\end{matrix} & \begin{matrix}0\\2\\0\end{matrix} & \begin{matrix}0\\0\\3\end{matrix} \end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$	0
x	$M(D)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix} \begin{matrix}0\\0\\0\end{matrix} & \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \begin{matrix}0\\2\\0\end{matrix} & \begin{matrix}0\\0\\3\end{matrix} \end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$	1
x^2	$M(D)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix} \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}1\\0\\0\end{matrix} & \boxed{\begin{matrix}0\\2\\0\end{matrix}} & \begin{matrix}0\\0\\3\end{matrix} \end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix} = 2\begin{pmatrix}0\\1\\0\end{pmatrix}$	$2x$
x^3	$M(D)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix} \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}1\\0\\0\end{matrix} & \begin{matrix}0\\2\\0\end{matrix} & \boxed{\begin{matrix}0\\0\\3\end{matrix}} \end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\3\end{pmatrix} = 3\begin{pmatrix}0\\0\\1\end{pmatrix}$	$3x^2$

verify with the Differentiation map

$D(1) = 0$

$D(x) = 1$

$D(x^2) = 2x$

$D(x^3) = 3x^2$

7/15 3.C continued

Addition and scalar multiplication of matrices

3.35 Definition:

Matrix addition is the sum of two matrices of the same size, obtained adding the corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n} \end{pmatrix}$$

In other words, $A_{j,k} + C_{j,k} = (A+C)_{j,k}$ for each $j=1,\dots,m$ and $k=1,\dots,n$

3.36 The matrix of the sum of linear maps:

Suppose $S, T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

3.37 Definition:

Scalar multiplication of a matrix is the product of a scalar and a matrix, obtained by multiplying each entry in the matrix by the scalar:

$$\lambda\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$, for each $j=1,\ldots,m$ and $k=1,\ldots,\cancel{m}\, n$

3.38 The matrix of a scalar times a linear map

Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

3.40 $\dim \mathbb{F}^{m,n} = mn$

Let m and n be positive integers. Let $\mathbb{F}^{m,n}$ is the set of all $m \times n$ matrices with entries in $\mathbb{F}$. Then $\dim \mathbb{F}^{m,n} = mn$.

Proof: $\mathbb{F}^{m,n}$ is a vector space with respect to the operations of matrix addition and scalar multiplication of a matrix. Now, $\dim \mathbb{F}^{m,n} = mn$ because

$\mathbb{F}^{\substack{m,n \\ 2,2}}$

basis of $\mathbb{F}^{2,2}$

2 entries

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$n=2$ matrices

$\dim \mathbb{F}^{2,2} = \overset{m}{2} \cdot \overset{n}{2} = 4$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$m=2$ matrices

n entries

$$\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

$m \times n$

n matrices

m matrices

$$\begin{pmatrix} 0 & \cdots & 0 \\ 1 & 0 \cdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 \\ 0 & 1 \cdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \ldots \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix}$$

$\vdots$

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{pmatrix}, \ldots\ldots \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

$\rightarrow$ is a basis of $\mathbb{F}^{m,n}$ (which you can verify)

So $\dim \mathbb{F}^{m,n} = mn$

Matrix Multiplication

Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_m$ be a basis of W. Also let U be a vector space and $u_1, \ldots, u_p$ is a basis of U.

Let $T: U \rightarrow V$ and $S: V \rightarrow W$ be linear maps.
We will show: $M(ST) = M(S)M(T)$.

Suppose $M(S) = A$ and $M(T) = C$. For each $k = 1, \ldots, p$, we have

$$
\begin{aligned}
(ST)u_k &= S(Tu_k) \\
&= S\boxed{\left(\sum_{r=1}^{n} C_{r,k} v_r\right)} \text{ by Def. 3.32 of Axler} \\
&= \sum_{r=1}^{n} C_{r,k} S v_n \text{ since } S \text{ is linear} \\
&= \sum_{r=1}^{n} C_{r,k} \boxed{\left(\sum_{j=1}^{m} A_{j,r} v_j\right)} \text{ by Def. 3.32 of Axler} \\
&= \sum_{j=1}^{m} \left(\sum_{r=1}^{n} A_{j,r} C_{r,k}\right) w_j
\end{aligned}
$$

By definition 3.32 of Axler, $D = M(ST)$ is the $m \times p$ matrix with entry in row j, column k,

$$D_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

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3.41 Definition:

Let A be an $m \times n$ matrix and C be an $n \times p$ matrix.
Then AC is an $m \times p$ matrix with entry in row j, column k.

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

In other words,

$$AC = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & & \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r} C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r} C_{r,p} \end{pmatrix}$$

3.42 Example

$$\underset{3\times 2}{\overset{A}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}} \underset{2\times 4}{\overset{C}{\begin{pmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix}}} = \underset{3\times 4}{\overset{AC}{\begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 12 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix}}}$$

3.43 The matrix of the product of linear maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, Then

$$\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$$

Let A be an $m \times n$ matrix,

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix}$$

Then $A_{j,\cdot} = (A_{j,1} \cdots A_{j,n})$

$$A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

3.45 Example

Let $A = \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix}$. Then

$A_{2,\cdot} = (1\ 9\ 7)$ and $A_{\cdot,2} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

3.46 Example

Let $A = (3\ \ 4)$ and $C = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$. Then

$AC = (3\ 4)\begin{pmatrix} 6 \\ 2 \end{pmatrix} = (26) = 26$

$(AC)_{1,1} = 26$ $\quad (AC)_{1,\cdot} = 26$ $\quad (AC)_{\cdot,1} = 26$

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3.47 Entry of matrix product equals row times column

Suppose A is a $m \times n$ matrix and C is an $n \times p$ matrix.
Then $(AC)_{j,k} = A_{j,\cdot} \, C_{\cdot,k}$

Proof: $(AC)_{j,k} = \left(\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix} \right)_{j,k}$

$$= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r} C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r} C_{r,p} \end{pmatrix}_{j,k}$$

$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

$$A_{j,\cdot} C_{\cdot,k} = (A_{j,1} \cdots A_{j,n}) \begin{pmatrix} C_{1,k} \\ \vdots \\ C_{n,k} \end{pmatrix}$$

$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

Therefore, $(AC)_{j,k} = A_{j,\cdot} \, C_{\cdot,k}$ ☒

3.50 Example

$$\underset{\substack{A \\ \text{in Ex. 3.42}}}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}} \underset{\substack{C_{\cdot,2} \\ \text{in Ex.} \\ 3.42}}{\begin{pmatrix} 5 \\ 1 \end{pmatrix}} = \underset{\substack{(AC)_{\cdot,2} \\ \text{in} \\ \text{Ex. 42}}}{\begin{pmatrix} 7 \\ 19 \\ 31 \end{pmatrix}}$$

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Ex 3.42

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$C = \begin{pmatrix} 6 & 9 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix}$$

$C_{\bullet,2}$

$$AC = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix}$$

$(AC)_{\bullet,2}$

$$(AC)_{\bullet,2} = AC_{\bullet,2}$$

3.52 Linear combination of columns

Suppose A is an $m \times n$ matrix and $C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an $n \times 1$ matrix.

Then
$$AC = c_1 A_{\bullet,1} + \dots + c_n A_{\bullet,n}$$

Proof:
$$c_1 A_{\bullet,1} + \dots + c_n A_{\bullet,n} = c_1 \begin{pmatrix} A_{1,1} \\ \vdots \\ A_{m,1} \end{pmatrix} + \dots + c_n \begin{pmatrix} A_{1,n} \\ \vdots \\ A_{m,n} \end{pmatrix}$$
$$= \begin{pmatrix} c_1 A_{1,1} \\ \vdots \\ c_1 A_{m,1} \end{pmatrix} + \dots + \begin{pmatrix} c_n A_{1,n} \\ \vdots \\ c_n A_{m,n} \end{pmatrix}$$
$$= \begin{pmatrix} c_1 A_{1,1} + \dots + c_n A_{1,n} \\ \vdots \\ c_1 A_{m,1} + \dots + c_n A_{m,n} \end{pmatrix}$$

$$AC = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,n} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1}C_1 + \cdots + A_{1,n}C_n \\ \vdots \\ A_{m,1}C_1 + \cdots + A_{m,n}C_n \end{pmatrix}$$

Therefore, $AC = C_1 A_{\cdot,1} + \cdots + C_n A_{\cdot,n}$.

Similar exercise 3.C.11 of Axler.