3C Matrices

Defn 3.30 Let m and n denote positive integers.
An $m \times n$ matrix A is a rectangular array of elements of $\mathbb{F}$, with m rows, and n columns:

$$\begin{array}{cc} A = \begin{pmatrix} A_{1,1} & A_{1,k} & \cdots & A_{1,n} \\ \vdots & \vdots & & \vdots \\ A_{j,1} & A_{j,k} & \cdots & A_{j,n} \\ \vdots & \vdots & & \vdots \\ A_{m,1} & A_{m,k} & & A_{m,n} \end{pmatrix} & \begin{array}{c} \\ \\ j\text{th row} \\ \\ \\ \end{array} \\ \quad\quad\quad k\text{th column} & \end{array}$$

The notation $A_{j,k}$ denotes the entry in row j column k of A.

row number

column number

Eg 3.31 If $A = \begin{pmatrix} 8 & 4 & 5-3i \\ 1 & 9 & 7 \end{pmatrix}$, then $A_{1,1} = 8$ $A_{1,2} = 4$ $A_{1,3} = 5-3i$
$A_{2,1} = 1$ $A_{2,2} = 9$ $A_{2,3} = 7$

Defn 3.32 Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \cdots, v_n$ is a basis of V and $w_1, \cdots, w_m$ is a basis of W.
The matrix of T with respect to these bases is the $m \times n$ matrix

$$M(T, (v_1, \cdots, v_n), (w_1, \cdots, w_n))$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k} w_1 + \cdots + A_{m,k} w_m.$$

If the bases are understood, then we can just write $M(T)$ denote the matrix of T with respect to the bases.

kth column of $M(T)$

$$M(T) = \begin{pmatrix} A_{1,1} & \cdots & \boxed{A_{1,k}} & \cdots & A_{1,n} \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix}$$

For any $k = 1 .. n$, we have

$$T v_k = \underbrace{\boxed{(w_1 \cdots w_m)}}_{1 \times m} \overset{k\text{th column of } \mathcal{M}(T)}{\underbrace{\boxed{\begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}}}_{m \times 1}}$$

identify all 1×1 matrices by their one entry

$$= \boxed{\left(A_{1,k} w_1 + \cdots + A_{m,k} w_m \right)} \quad 1 \times 1$$

$$= A_{1,k} w_1 + \cdots + A_{m,k} w_m$$

$$= \sum_{j=1}^{m} A_{j,k} w_j$$

Eg 3.33 Define $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ by

$$T(x,y) = (x + 3y, 2x + 5y, 7x + 9y)$$

Find the matrix of T with respect to the standard bases of $\mathbb{F}^2, \mathbb{F}^3$.

In other words, find

$$\mathcal{M}(T) = \mathcal{M}\Big(T, \underbrace{\boxed{(\overset{v_1}{(1,0)}, \overset{v_2}{(0,1)})}}_{\text{standard bases } \mathbb{F}^2}, \underbrace{\boxed{(\overset{w_1}{(1,0,0)}, \overset{w_2}{(0,1,0)}, \overset{w_3}{(0,0,1)})}}_{\text{standard bases } \mathbb{F}^3}\Big)$$

soln: We have

$$T(\overset{v_1}{1,0}) = (1,2,7)$$

$$T(\overset{v_2}{0,1}) = (3,5,9)$$

The matrix of T with respect to the standard bases is

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$

$$= \left(T(1,0),\ T(0,1) \right)$$

$$T(1,0) = T v_1$$

$$= \sum_{j=1}^{3} A_{j,1} w_j$$

$$= A_{1,1} w_1 + A_{2,1} w_2 + A_{3,1} w_3$$

$$= 1(1,0,0) + 2(0,1,0) + 7(0,0,1)$$

$$= (1,2,7)$$

$$T(0,1) = T v_2$$

$$= \sum_{j=1}^{3} A_{j,2} w_j$$

$$= A_{1,2} w_j + A_{2,2} w_j + A_{3,2} w_j$$

$$= 3(1,0,0) + 5(0,1,0) + 9(0,0,1)$$

$$= (3,5,9)$$

Eg 3.34

Let $1, x, x^2, \ldots, x^m$ be the standard basis of $\mathcal{P}_m(\mathbb{F})$.
Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by

$$Dp = p'$$

Find the matrix of D with respect to the standard bases of $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$.

In other words, find

$$\mathcal{M}(D) = \mathcal{M}\Big(D, \underbrace{\boxed{(1, x, x^2, x^3)}}_{\text{standard bases of } \mathcal{P}_3(\mathbb{R})}, \underbrace{\boxed{(1, x, x^2)}}_{\text{standard bases of } \mathcal{P}_2(\mathbb{R})}\Big)$$

soln. Basis of $\mathcal{P}_3(\mathbb{R})$ is $1, x, x^2, x^3$

Basis of $\mathcal{P}_2(\mathbb{R})$ is $1, x, x^2$

For all positive integers n,

$$D(x^n) = (x^n)' = n x^{n-1}$$

Input in $P_3(\mathbb{R})$		Corresponding vector in $\mathbb{F}^4$
1	$\longleftrightarrow$	$(1,0,0,0)$
x	$\longleftrightarrow$	$(0,1,0,0)$
x^2	$\longleftrightarrow$	$(0,0,1,0)$
x^3	$\longleftrightarrow$	$(0,0,0,1)$

Output in $P_2(\mathbb{R})$		Corresponding vector in $\mathbb{F}^3$
$D(1)=0$	$\longleftrightarrow$	$(0,0,0)$
$D(x)=1$	$\longleftrightarrow$	$(1,0,0)$
$D(x^2)=2x$	$\longleftrightarrow$	$(0,2,0)$
$D(x^3)=3x^2$	$\longleftrightarrow$	$(0,0,3)$

$$M(D)=\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}$$

Verify with the differentiation map

Input in $P_3(\mathbb{R})$		Output in $P_2(\mathbb{R})$	
1	$M(D)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}\boxed{0}&1&0&0\\\boxed{0}&0&2&0\\\boxed{0}&0&0&3\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$	0	$D(1)=0$
x	$M(D)\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0&\boxed{1}&0&0\\0&\boxed{0}&2&0\\0&\boxed{0}&0&3\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$	1	$D(x)=1$
x^2	$M(D)\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0&1&\boxed{0}&0\\0&0&\boxed{2}&0\\0&0&\boxed{0}&3\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix}=2\begin{pmatrix}0\\1\\0\end{pmatrix}$	$2x$	$D(x^2)=2x$
x^3	$M(D)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0&1&0&\boxed{0}\\0&0&2&\boxed{0}\\0&0&0&\boxed{3}\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\3\end{pmatrix}=3\begin{pmatrix}0\\0\\1\end{pmatrix}$	$3x^2$	$D(x^3)=3x^2$

$\boxed{\tfrac{1}{2}}$

7/15/2019 3C continued: Matrices

Addition and Scalar multiplication of Matrices

Defn 3.35 **Matrix Addition** is the sum of two matrices of the same size, obtained by adding the corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n} \end{pmatrix}$$

In other words, $A_{j,k} + C_{j,k} = (A+C)_{j,k}$ for each $j=1,\dots,m$ and $k=1,\dots,n$.

Defn 3.36 **The matrix of the sum of linear maps**

Suppose $S, T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$

Defn 3.37 **Scalar Multiplication of a matrix** is the product of a scalar and a matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$ for each $j=1,\dots,m$ and $k=1,\dots,n$.

Defn 3.38 **The matrix of a scalar times a linear map**

Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$

3.40 $\dim \mathbb{F}^{m,n} = mn$

Let m and n be positive integers. Let $\mathbb{F}^{m,n}$ is the set of all $m \times n$ matrices with entries in $\mathbb{F}$

Then $\dim \mathbb{F}^{m,n} = mn$.

proof: $\mathbb{F}^{m,n}$ is a vector space with respect to the operations of matrix addition and scalar multiplication of a matrix.

Now, $\dim \mathbb{F}^{m,n} = mn$ because →

n matrices

m entries

$$\left(\begin{smallmatrix}1&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix}0&\cdots&1\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{smallmatrix}\right)$$

n matrices

$$\left(\begin{smallmatrix}0&\cdots&0\\ 1&0&\vdots\\ 0&\cdots&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&\cdots&0\\ 0&1&\vdots\\ 0&\cdots&0\end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix}0&\cdots&0\\ 0&\ddots&1\\ 0&\cdots&0\end{smallmatrix}\right)$$

n matrices

m matrices

$\vdots$

$$\left(\begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 1&\cdots&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&1\cdots&0\end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&1\end{smallmatrix}\right)$$

$m \times n$

m rows

n columns

$\hookrightarrow$ is a basis of $\mathbb{F}^{m,n}$ (which you can verify)

So, $\dim \mathbb{F}^{m,n} = mn$

$\mathbb{F}^{m,n}_{2,2}$

basis of $\mathbb{F}^{2,2}$

2 entries

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

n=2 matrices

M.n

m=2 matrices

$$\dim \mathbb{F}^{2,2} = m\cdot n = 2\cdot 2 = 4$$

Matrix Multiplication

Let $v_1, \dots, v_n$ be a basis of V and $w_1, \dots, w_m$ be a basis of W. Also let U be a vector space and $u_1, \dots, u_p$ is a basis of U.

Let $T: U \to V$ and $S: V \to W$ be linear maps.

We will show: $M(ST) = M(S)M(T)$

Suppose $M(S) = A$ and $M(T) = C$. For each $k = 1, \dots, p$, we have ~~$(ST)u_k = S\left(\sum_{r=1}^{n} C_{r,k} v_r\right)$~~ by Defn 3.32 of Axler

$$\begin{aligned}(ST)u_k &= S(Tu_k)\\ &= S\boxed{\left(\sum_{r=1}^{n} C_{r,k} v_r\right)}\\ &= \sum_{r=1}^{n} C_{r,k} Sv_r \quad \text{since S is linear}\\ &= \sum_{r=1}^{n} C_{r,k}\boxed{\left(\sum_{j=1}^{m} A_{j,r} w_j\right)} \quad \text{by Defn 3.32 of Axler}\\ &= \sum_{j=1}^{m}\left(\sum_{r=1}^{n} A_{j,r} C_{r,k}\right) w_j\end{aligned}$$

By Definition of 3.32 of Axler, "D" = M(ST) is the $m \times p$ matrix with entry in row j, column k,

$$D_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

Defn 3.41 Let A be an $m \times n$ matrix with entry row j, column k:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

In Other words,

$$AC = \begin{pmatrix}A_{1,1}&\cdots&A_{1,n}\\ \vdots&\ddots&\vdots\\ A_{m,1}&\cdots&A_{m,n}\end{pmatrix}\begin{pmatrix}C_{1,1}&\cdots&C_{1,n}\\ \vdots&\ddots&\vdots\\ C_{m,1}&\cdots&C_{m,n}\end{pmatrix}$$

$$= \begin{pmatrix}\sum_{r=1}^{n} A_{1,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r} C_{r,p}\\ \vdots & & \\ \sum_{r=1}^{n} A_{m,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r} C_{r,p}\end{pmatrix}$$

Eg 3.42

$$\underset{3\times 2}{\overset{A}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}} \underset{2\times 4}{\overset{C}{\begin{pmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix}}} = \underset{3\times 4}{\overset{AC}{\begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix}}}$$

Defn 3.43 The matrix of the product of linear maps

If $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$, Then

$$M(ST) = M(S)M(T)$$

Let A be an $m \times n$ matrix,

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix}$$

Then $A_{j,\cdot} = (A_{j,1} \cdots A_{j,n})$

$$A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

Eg 3.45 Let $A = \begin{pmatrix} 8 & \boxed{4} & 5 \\ \boxed{1} & \boxed{9} & \boxed{7} \end{pmatrix}$. Then

$A_{2,\cdot} = (1\ 9\ 7)$ and $A_{\cdot,2} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

Eg 3.46 Let $A = (3\ 4)$ and $C = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$. Then

$AC = (3\ 4)\begin{pmatrix} 6 \\ 2 \end{pmatrix} = (26) = 26$

$(AC)_{1,1} = 26 \qquad (AC)_{1,\cdot} = 26 \qquad (AC)_{\cdot,1} = 26$

Defn 3.47 Entry of matrix product equals row times column

Suppose A is a $m \times n$ matrix and C is an $m \times n$ matrix. Then,

$$(AC)_{j,k} = A_{j,\cdot}\, C_{\cdot,k}.$$

proof.

$$(AC)_{j,k} = \left(\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix} \right)_{j,k}$$

$$= \left(\begin{pmatrix} \sum_{r=1}^{n} A_{1,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r} C_{r,p} \\ \vdots & & \vdots \\ \sum_{r=1}^{n} A_{m,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r} C_{r,p} \end{pmatrix} \right)_{j,k}$$

$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

$$A_{j,\cdot} C_{\cdot,k} = (A_{j,1} \cdots A_{j,n}) \begin{pmatrix} C_{1,k} \\ \vdots \\ C_{n,k} \end{pmatrix}$$

$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

Therefore, $(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k}$ ∎

Eg 3.50

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \\ 31 \end{pmatrix}$$

A in Ex 3.42 $\qquad C_{\cdot,2}$ in Ex 3.42 $\qquad (AC)_{\cdot,2}$ in Ex 3.42

Eg 3.42

$A = \boxed{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}$ A

$C = \begin{pmatrix} 6 & \boxed{5} & 4 & 3 \\ 2 & \boxed{1} & 0 & -1 \end{pmatrix}$ $C_{\cdot,2}$

$AC = \begin{pmatrix} 10 & \boxed{7} & 4 & 1 \\ 26 & \boxed{19} & 12 & 5 \\ 42 & \boxed{31} & 20 & 9 \end{pmatrix}$ $(AC)_{\cdot,2}$

$(AC)_{\cdot,2} = AC_{\cdot,2}$

Defn 3.52 Linear Combination of columns

Suppose A is an $m \times n$ matrix and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is a $n \times 1$ matrix. Then

$$Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$$

proof: $c_1 A_{\cdot,1} + \cdots c_n A_{\cdot,n} = c_1 \begin{pmatrix} A_{1,1} \\ \vdots \\ A_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} A_{1,n} \\ \vdots \\ A_{m,n} \end{pmatrix}$

$$= \begin{pmatrix} c_1 A_{1,1} \\ \vdots \\ c_1 A_{m,1} \end{pmatrix} + \cdots + \begin{pmatrix} c_n A_{1,n} \\ \vdots \\ c_n A_{m,n} \end{pmatrix}$$

→

$$= \begin{pmatrix} c_1 A_{1,1} + \cdots + c_n A_{1,n} \\ \vdots \\ c_1 A_{m,1} + \cdots + c_n A_{m,n} \end{pmatrix}$$

$$AC = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1} c_1 + \cdots + A_{1,n} c_n \\ \vdots \\ A_{m,1} c_1 + \cdots + A_{m,n} c_n \end{pmatrix}$$

Therefore, $AC = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$. ☒

(Similar to excercise: 3.C.11 of Axler)