3.C Matrices

3.30 Def.

Let m & n denote positive integers

An $m \times n$ matrix A is a rectangular array of elements of $\mathbb{F}$, w/ m rows & n columns:

$$A = \begin{array}{c} \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix} \quad j^{th} \text{ row} \\ k^{th} \text{ column} \end{array}$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A

row # column #

3.31 Example

If $A = \begin{pmatrix} 8 & 4 & 5-3i \\ 1 & 9 & 7 \end{pmatrix}$, then $A_{1,1} = 8$ $A_{1,2} = 4$ $A_{1,3} = 5-3i$

$A_{2,1} = 1$ $A_{2,2} = 9$ $A_{2,3} = 7$

3.32 Def.

Suppose $T \in \mathcal{L}(V,W)$ & $v_1, \ldots, v_n$ is a basis of V & $w_1, \ldots, w_m$ is a basis of W. The matrix of T w/ respect to these bases is the $m \times n$ matrix

$$M(T(v_1, \ldots, v_n), (w_1, \ldots, w_m))$$

whose entries $A_{j,k}$ are defined by $Tv_k = A_{1,k} w_1 + \ldots + A_{m,k} w_m$

If the basis are understood, then we can just write $M(T)$ to denote the matrix of T w/ respect to the bases.

$$M(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix}$$

For any $k = 1, \ldots, n$, we have

$$Tv_k = \underbrace{(w_1, \ldots, w_m)}_{1 \times m} \underbrace{\begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}}_{m \times 1}$$

Identify all 1×1 matrices by their one entry

$$= \underbrace{(A_{1,k} w_1 + \ldots + A_{m,k} w_m)}_{1 \times 1}$$

$$= A_{1,k} w_1 + \ldots + A_{m,k} w_m$$

$$= \sum_{j=1}^{m} A_{j,k} w_j$$

3.33 Example Define $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ by $T(x,y) = (x+3y,\ 2x+5y,\ 7x+9y)$

Find the matrix of T w/ respect to the standard bases of $\mathbb{F}^2$ & $\mathbb{F}^3$.

In other words, find

$$M(T) = M\Big(T, \underbrace{(\overset{v_1}{(1,0)}, \overset{v_2}{(0,1)})}_{\text{standard basis of } \mathbb{F}^2}, \underbrace{(\overset{w_1}{(1,0,0)}, \overset{w_2}{(0,1,0)}, \overset{w_3}{(0,0,1)})}_{\text{standard basis of } \mathbb{F}^3}\Big)$$

Soln: we have

$T(\overset{v_1}{1,0}) = (1,2,7)$

$T(\overset{v_2}{0,1}) = (3,5,9)$

The matrix of T w/ respect to the standard bases is

$$M(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix} = \big(T(1,0)\ T(0,1)\big)$$

$$T(1,0) = Tv_1 = \sum_{j=1}^{3} A_{j,1} w_j = A_{1,1}w_1 + A_{2,1}w_2 + A_{3,1}w_3 = 1(1,0,0) + 2(0,1,0) + 7(0,0,1) = (1,2,7)$$

$$T(0,1) = Tv_2 = \sum_{j=1}^{3} A_{j,2} w_j = A_{1,2}w_1 + A_{2,2}w_2 + A_{3,2}w_3$$
$$= 3(1,0,0) + 5(0,1,0) + 9(0,0,1)$$
$$= (3,5,9)$$

3.39 Example

Let $1, x, x^2, \dots, x^m$ be the standard basis of $P_m(\mathbb{F})$

Suppose $D \in \mathcal{L}(P_3(\mathbb{R}), P_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find the matrix of D w/ respect to the standard basis of $P_3(\mathbb{R})$ & $P_2(\mathbb{R})$. In other words, find

$$M(D) = M\Big(D, \underbrace{(1, x, x^2, x^3)}_{\text{standard basis of } P_3(\mathbb{R})}, \underbrace{(1, x, x^2)}_{\text{S.B. of } P_2(\mathbb{R})}\Big)$$

Soln: Basis of $P_3(\mathbb{R})$ is $1, x, x^2, x^3$

Basis of $P_2(\mathbb{R})$ is $1, x, x^2$

For all positive integers n, $D(x^n) = (x^n)' = nx^{n-1}$

Input in $P_3(\mathbb{R})$		corresponding vector in $\mathbb{F}^4$	output in $P_2(\mathbb{R})$		corr. vector in $\mathbb{F}^3$
1	$\overset{1}{\longleftrightarrow}$	(1,0,0,0)	$D(1) = 0$	$\longleftrightarrow$	(0,0,0)
x	$\overset{2}{\longleftrightarrow}$	(0,1,0,0)	$D(x) = 1$	$\longleftrightarrow$	(1,0,0)
x^2	$\longleftrightarrow$	(0,0,1,0)	$D(x^2) = 2x$	$\longleftrightarrow$	(0,2,0)
x^3	$\longleftrightarrow$	(0,0,0,1)	$D(x^3) = 3x^2$	$\longleftrightarrow$	(0,0,3)

$$M(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Input in $P_3(\mathbb{R})$		output in $P_2(\mathbb{R})$	verify w/ the differentiation map
1	$M(D)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix} 0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$	0	$D(1)=0$
x	$M(D)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix} 0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$	1	$D(x)=1$
x^2	$M(D)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix} 0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix} = 2\begin{pmatrix}0\\1\\0\end{pmatrix}$	$2x$	$D(x^2)=2x$
x^3	$M(D)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix} 0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\3\end{pmatrix} = 3\begin{pmatrix}0\\0\\1\end{pmatrix}$	$3x^2$	$D(x^3)=3x^2$

7/15/19
Mon.
week 4

3C continued

Addition and scalar multiplication of matrices

3.35 Def.

Matrix addition is the sum of two matrices of the same size, obtained adding the corresponding entries in the matrices

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n} \\ \vdots & & \vdots \\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n} \end{pmatrix}$$

In other words, $A_{j,k} + C_{j,k} = (A+C)_{j,k}$ for each $j = 1, \ldots, m$ & $k = 1, \ldots, n$

3.36 The matrix of the sum of linear maps

Suppose $S, T \in \mathcal{L}(V, W)$. Then $M(S+T) = M(S) + M(T)$

3.37 Def

scalar multiplication of a matrix is the product of a scalar and a matrix, obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & & \lambda A_{m,n} \end{pmatrix}$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$, for each $j = 1, \ldots, m$ & $k = 1, \ldots, n$

3.38 The matrix of a scalar times a linear map

Suppose $\lambda \in \mathbb{F}$ & $T \in \mathcal{L}(V, W)$. Then $M(\lambda T) = \lambda M(T)$

3.40 $\dim \mathbb{F}^{m,n} = mn$

Let m & n be positive integers. Let $\mathbb{F}^{m,n}$ is the set of all $m \times n$ matrices w/ entries in $\mathbb{F}$. Then $\dim \mathbb{F}^{m,n} = mn$

Proof: $F^{m,n}$ is a vector space with respect to the operations of matrix addition & scalar multiplication of a matri…

Now, dim $F^{m,n} = mn$ because

m matrices

$$\underbrace{\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}_{n \text{ matrices (across)}}$$

m×n
m rows
n columns

$$\underbrace{\begin{pmatrix} 0 & \cdots & 0 \\ 1 & 0 & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 \\ 0 & 1 & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & 1 \\ 0 & \cdots & 0 \end{pmatrix}}_{n \text{ matrices}}$$

$\vdots$

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 1 \cdots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

is a basis of $F^{m,n}$ so $\dim F^{m,n} = mn$

Matrix Multiplication

Let $v_1, \dots, v_n$ be a basis of V & $w_1, \dots, w_m$ be a basis of W. Also let U be a vector space & $u_1, \dots, u_p$ is a basis of U.

Let $T: U \to V$ & $S: V \to W$ be linear maps

We will show: $M(ST) = M(S)M(T)$

Suppose $M(S)$ & $M(T) = C$. For each $k = 1, \dots, p$ we have $(ST)u_k = S(Tu$…

$$(ST)u_k = S\left(\sum_{r=1}^{n} C_{r,k} v_r\right) \text{ by def. 3.32 of Axler}$$

$$= \sum_{r=1}^{n} C_{r,k} S v_r \quad \text{since } S \text{ is linear}$$

$$= \sum_{r=1}^{n} C_{r,k} \left(\sum_{j=1}^{m} A_{j,r} w_j\right) \text{ by def 3.32 of Axler}$$

$$= \sum_{j=1}^{m} \left(\sum_{r=1}^{n} A_{j,r} C_{r,k}\right) w_j$$

By def 3.32 of Axler, $M(ST)$ is the $m \times p$ matrix w/ entry in row j, column k, $D_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$

3.41 Def.

Let A be an $m \times n$ matrix & C be an $n \times p$ matrix. Then AC is an $m \times p$ matrix w/ entry in row j, column k:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

In other words,

$$AC = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r}C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r}C_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r}C_{r,p} \end{pmatrix}$$

3.42 Example

$$\underset{3 \times 2}{\overset{A}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}} \underset{2 \times 4}{\overset{C}{\begin{pmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix}}} = \underset{3 \times 4}{\overset{AC}{\begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix}}}$$

3.43 The matrix of product of linear maps

If $T \in \mathcal{L}(U,V)$ & $S \in \mathcal{L}(V,W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\,\mathcal{M}(T)$

Let A be an $m \times n$ matrix,

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{j,k} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & & \vdots \\ A_{j,1} & & A_{j,k} & & A_{j,n} \\ \vdots & & \vdots & \ddots & \vdots \\ A_{m,1} & & A_{m,k} & & A_{m,n} \end{pmatrix}$$

Then

$$A_{j,\bullet} = (A_{j,1} \cdots A_{j,n})$$

$$A_{\bullet,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

3.45 Example

Let $A = \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix}$. Then

$A_{2,\cdot} = (1, 9, 7)$ & $A_{\cdot,2} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

3.46 Example

Let $A = (3 \;\; 4)$ & $C = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$. Then $AC = (3 \;\; 4)\begin{pmatrix} 6 \\ 2 \end{pmatrix} = (26) = 26$

$(AC)_{1,1} = 26 \quad (AC)_{1,\cdot} = 26 \quad (AC)_{\cdot,1} = 26$

3.47 Entry of matrix product equals row times column

Suppose A is a $m \times n$ matrix & C is an $n \times p$ matrix. Then $(AC)_{j,k} = A_{j,\cdot}C_{\cdot,k}$

Proof:

$$(AC)_{j,k} = \left(\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix} \right)_{j,k}$$

$$= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r}C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r}C_{r,k} & \cdots & \sum_{r=1}^{n} A_{m,r}C_{r,p} \end{pmatrix}_{j,k}$$

$$= \sum_{r=1}^{n} A_{j,r}C_{r,k}$$

$$A_{j,\cdot}C_{\cdot,k} = (A_{j,1} \cdots A_{j,n}) \begin{pmatrix} C_{1,k} \\ \vdots \\ C_{n,k} \end{pmatrix} = \sum_{r=1}^{n} A_{j,r}C_{r,k}$$

Therefore, $(AC)_{j,k} = A_{j,\cdot}C_{\cdot,k}$

3.50 Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \\ 31 \end{pmatrix}$$

A in example 3.42 $C_{\cdot,2}$ in example 3.42 $(AC)_{\cdot,2}$ in example

Example 3.42

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ $C = \begin{pmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix}$

$AC = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix}$ $(AC)_{\cdot,2}$

$(AC)_{\cdot,2} = AC_{\cdot,2}$

3.52 Linear Combo of columns

Suppose A is an $m \times n$ matrix & $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an $n \times 1$ matrix. Then

$AC = c_1 A_{\cdot,1} + \ldots + c_n A_{\cdot,n}$

proof:

$$c_1 A_{\cdot,1} + \ldots + c_n A_{\cdot,n} = c_1 \begin{pmatrix} A_{1,1} \\ \vdots \\ A_{m,1} \end{pmatrix} + \ldots + c_n \begin{pmatrix} A_{1,n} \\ \vdots \\ A_{m,n} \end{pmatrix} = \begin{pmatrix} c_1 A_{1,1} \\ \vdots \\ c_1 A_{m,1} \end{pmatrix} + \ldots + \begin{pmatrix} c_n A_{1,n} \\ \vdots \\ c_n A_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 A_{1,1} + \ldots + c_n A_{1,n} \\ \vdots \\ c_1 A_{m,1} + \ldots + c_n A_{m,n} \end{pmatrix}$$

$$AC = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} A_{1,1} c_1 + \ldots + A_{1,n} c_n \\ \vdots \\ A_{m,1} c_1 + \ldots + A_{m,n} c_n \end{pmatrix}$$

Therefore, $AC = c_1 A_{\cdot,1} + \ldots + c_n A_{\cdot,n}$

similar exercise 3.C.11 of Axler