07-10-19

3.C Matrices

3.30 Definition

Let m and n denote positive integers. An $m \times n$ matrix A is a rectangular array of elements of $\mathbb{F}$, with m rows and n columns:

$$\begin{array}{cc} \begin{array}{c} k^{th} \text{ column} \end{array} & \\ A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix} & j^{th} \text{ row} \end{array}$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A

row # column #

3.31 Example

If $A = \begin{pmatrix} 8 & 4 & 5-3i \\ 1 & 9 & 7 \end{pmatrix}$

then $A_{1,1} = 8$ $A_{1,2} = 4$ $A_{1,3} = 5-3i$

$A_{2,1} = 1$ $A_{2,2} = 9$ $A_{2,3} = 7$

3.32 Definition

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. The matrix of T with respect to these bases is the $m \times n$ matrix

$$\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k} w_1 + \cdots + A_{m,k} w_m$$

If the bases are understood, the we can just write $\mathcal{M}(T)$ denote the matrix of T with respect to the bases.

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,k} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix} \quad k^{th} \text{ column of } \mathcal{M}(T)$$

For any $k = 1, \dots, n$ we have

$$\underset{1\times m}{Tv_k = (w_1, \cdots, w_m)} \underset{m \times 1}{\begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}}$$ kth column of $M(T)$

Identify all 1×1 matrices by their one entry

$$= \underset{1\times 1}{(A_{1,k}w_1 + \cdots + A_{m,k}w_m)}$$

$$= A_{1,k}w_1 + \cdots + A_{m,k}w_m$$

$$= \sum_{j=1}^{m} A_{j,k}w_j$$

3.33 Example

Define $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ by

$$T(x,y) = (x+3y,\ 2x+5y,\ 7x+9y)$$

Find the matrix of T with respect to the standard bases of $\mathbb{F}^2$ and $\mathbb{F}^3$

In other words, find

$$MT = M\Big(T\ \underbrace{(1,0),(0,1)}_{\text{standard base of } \mathbb{F}^2}\Big), \Big(\underbrace{(1,0,0),(0,1,0),(0,0,1)}_{\text{standard base of } \mathbb{F}^3}\Big)$$

Solution: we have

$$T(1,0) = (1,2,7)$$
$$T(0,1) = (3,5,9)$$

The matrix of T with respect to the standard bases is

$$M(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$

$$= \big(T(1,0)\ \ T(0,1)\big)$$

$$T(1,0) = Tv_1$$
$$= \sum_{j=1}^{3} A_{j,1}w_j$$
$$= A_{1,1}w_1 + A_{2,1}w_2 + A_{3,1}w_3$$
$$= 1(1,0,0) + 2(0,1,0) + 7(0,0,1)$$
$$= (1,2,7)$$

$$T(0,1) = Tv_2$$
$$= \sum_{j=1}^{3} A_{j,2}w_j$$
$$= A_{1,2}w_1 + A_{2,2}w_2 + A_{3,2}w_3$$
$$= 3(1,0,0) + 5(0,1,0) + 9(0,0,1)$$
$$= (3,5,9)$$

3.34 Example

Let $1, x, x^2, \ldots, x^m$ be the standard basis of $P_m(\mathbb{F})$

Suppose $D \in \mathcal{L}(P_3(\mathbb{R}), P_2(\mathbb{R}))$ is the differentiation map defined by

$$Dp = p'$$

Find the matrix of D with respect to the standard bases of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$

In other words, find

$$M(D) = M(D, \underbrace{(1, x, x^2, x^3)}_{\substack{\text{standard bases} \\ \text{of} \\ P(\mathbb{R}^3)}}, \underbrace{(1, x, x^2)}_{\substack{\text{standard} \\ \text{bases} \\ \text{of} \\ P(\mathbb{R}^2)}})$$

Solution: Basis of $P_3(\mathbb{R})$ is $1, x, x^2, x^3$

Basis of $P_2(\mathbb{R})$ is $1, x, x^2$

For all positive integers n,

$$D(x^n) = (x^n)' = nx^{n-1}$$

Input in $P_3(\mathbb{R})$		Corresponding vector in $\mathbb{F}^4$	Output in $P_2(\mathbb{R})$		Corresponding vector in $\mathbb{F}^3$
1	$\longleftrightarrow$	$(1,0,0,0)$	$D(1)=0$	$\longleftrightarrow$	$(0,0,0,0)$
x	$\longleftrightarrow$	$(0,1,0,0)$	$D(x)=1$	$\longleftrightarrow$	$(1,0,0,0)$
x^2	$\longleftrightarrow$	$(0,0,1,0)$	$D(x^2)=2x$	$\longleftrightarrow$	$(0,2,0,0)$
x^3	$\longleftrightarrow$	$(0,0,0,1)$	$D(x^3=3x^2$	$\longleftrightarrow$	$(0,0,3,0)$

$$M(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Input in $P_3(\mathbb{R})$ — verify with the Differentiation map

1: $M(D)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix} \boxed{0} & 1 & 0 & 0 \\ \boxed{0} & 0 & 2 & 0 \\ \boxed{0} & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} = 0$ $\qquad D(1) = 0$

x: $M(D)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix} 0 & \boxed{1} & 0 & 0 \\ 0 & \boxed{0} & 2 & 0 \\ 0 & \boxed{0} & 0 & 3 \end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} = 1$ $\qquad D(x) = 1$

x^2: $M(D)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix} 0 & 1 & \boxed{0} & 0 \\ 0 & 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{0} & 3 \end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix} = 2\begin{pmatrix}0\\1\\0\end{pmatrix} = 2x$ $\qquad D(x^2) = 2x$

x^3: $M(D)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \boxed{0} \\ 0 & 0 & 2 & \boxed{0} \\ 0 & 0 & 0 & \boxed{3} \end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\3\end{pmatrix} = 3\begin{pmatrix}0\\0\\1\end{pmatrix} = 3x^2$ $\qquad D(x^3) = 3x^2$

07-15-19

Addition and scalar multiplication of matrices

3.55 Definition

Matrix addition is the sum of two matrices of the same size obtained adding the corresponding entries in the matrices

$$\underbrace{\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}}_{\in \mathbb{F}^{m,n}} + \underbrace{\begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix}}_{\in \mathbb{F}^{m,n}} = \underbrace{\begin{pmatrix} A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n} \\ \vdots & & \vdots \\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n} \end{pmatrix}}_{\in \mathbb{F}^{m,n}}$$

In other words, $A_{j,k} + C_{j,k} = (A+C)_{j,k}$ for each $j=1,\dots,m$ and $k=1,\dots,n$

3.36 The matrix of the sum of linear maps

Suppose $S, T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$

3.37 Definition

Scalar multiplication of a matrix is the product of a scalar and a matrix, obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$, for each $j=1,\dots,m$ and $k=1,\dots,m$

3.38 The matrix of a scalar times a linear map

Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$

3.40 $\dim \mathbb{F}^{m,n} = mn$

Let m and n be positive integers. Let $\mathbb{F}^{m,n}$ is the set of all $m \times n$ matrices with entries in $\mathbb{F}$

Then $\dim \mathbb{F}^{m,n} = mn$

Proof: $\mathbb{F}^{m,n}$ is a vector space with respect to the operations of matrix addition and scalar multiplication of a matrix

Now, $\dim \mathbb{F}^{m,n} = mn$ because

$$\underbrace{\overbrace{\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}^{n \text{ entries}}, \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}_{n \text{ matrices}}$$

$m \times n$

m rows

n columns

$$\text{m matrices}\left\{\begin{array}{l}\begin{pmatrix}0&\cdots&0\\1&0&\vdots\\\vdots&\ddots&\vdots\\0&\cdots&0\end{pmatrix},\begin{pmatrix}0&\cdots&0\\0&1&\vdots\\\vdots&\ddots&\vdots\\0&\cdots&0\end{pmatrix},\ldots,\begin{pmatrix}0&\cdots&0\\\vdots&\ddots&1\\\vdots&&\vdots\\0&\cdots&0\end{pmatrix}\\ \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{n matrices}}\\ \begin{pmatrix}0&\cdots&0\\\vdots&\ddots&\vdots\\1&\cdots&0\end{pmatrix},\begin{pmatrix}0&\cdots&0\\\vdots&\ddots&\vdots\\0&1\cdots&0\end{pmatrix},\ldots,\begin{pmatrix}0&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&1\end{pmatrix}\end{array}\right.$$

$\mathbb{F}^{m,n}_{2,2}$

basis of $\mathbb{F}^{m,n}_{2,n}$

$$\boxed{\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$n = 2$ matrices

$$\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$m = 2$ matrices

$\dim \mathbb{F}^{2,2} = m \cdot n$

$= 2 \cdot 2 = 4$

is a basis of $\mathbb{F}^{m,n}$ (which you can verify)

So $\dim \mathbb{F}^{m,n} = mn$

Matrix Multiplication

Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_m$ be a basis of W.

Also let U be a vector space and $u_1, \ldots, u_p$ is a basis of U.

Let $T: U \to V$ and $S: V \to W$ be linear maps

we will show: $\mathcal{M}(ST) = \mathcal{M}(S)\,\mathcal{M}(T)$

Suppose $\mathcal{M}(S) = A$ and $\mathcal{M}(T) = C$. For each $k = 1, \ldots, p$, we have

$(ST)u_k = S(Tu_k) = (ST)u_k = S\boxed{\left(\sum_{r=1}^{n} C_{r,k} v_r\right)}$ by Definition 3.32 of Axler

$= \sum_{r=1}^{n} C_{r,k} S v_r$ since S is linear

$= \sum_{r=1}^{n} C_{r,k} \boxed{\left(\sum_{j=1}^{m} A_{j,r} w_j\right)}$ by Definition 3.32 of Axler

$= \sum_{j=1}^{m}\left(\sum_{r=1}^{n} A_{j,r} C_{r,k}\right) w_j$

By Definition 3.32 of Axler, "D" $= \mathcal{M}(ST)$ is the $m \times p$ matrix with entry in row j, column k,

$$D_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

3.41 Definition

Let A be an $m \times n$ matrix and C be an $n \times p$ matrix. Then AC is an $m \times p$ matrix with entry r, row j, column k:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

In other words,

$$AC = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix} = \begin{pmatrix} \sum_{r=1}^{n} A_{1,r}C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r}C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r}C_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r}C_{r,p} \end{pmatrix}$$

3.42 Example

$$\underset{3\times 2}{\overset{A}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}} \underset{2\times 4}{\overset{C}{\begin{pmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix}}} = \underset{3\times 4}{\overset{AC}{\begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix}}}$$

3.43 The matrix of the product of linear maps

If $T \in \mathcal{L}(U,V)$ and $S \in (V,W)$, Then

$$\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$$

Let A be an $m \times n$ matrix,

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1n} \\ \vdots & & \vdots & & \vdots \\ A_{j1} & \cdots & A_{jk} & \cdots & A_{jn} \\ \vdots & & \vdots & & \vdots \\ A_{m1} & \cdots & A_{mk} & \cdots & A_{mn} \end{pmatrix}$$

Then

- $A_{j,\bullet} = (A_{j,1} \cdots A_{j,k} \cdots A_{j,n})$
- $A_{\bullet,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$

3.45 Example

Let $A = \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix}$ Then

$A_{2,\bullet} = (1\ 9\ 7)$ and $A_{\bullet,2} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

3.46 Example

Let $A = (3\ 4)$ and $C = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$. Then

$$AC = (3\ 4)\begin{pmatrix} 6 \\ 2 \end{pmatrix} = (26) = 26$$

$(AC)_{1,1} = 26$ $\quad$ $(AC)_{1,\bullet} = 26$ $\quad$ $(AC)_{\bullet,1} = 26$

3.47 Entry of matrix product equals row times column

Suppose A is an $m \times n$ matrix and C is an $n \times p$ matrix. Then

$$(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k}$$

Proof: $$(AC)_{j,k} = \left(\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix} \right)_{j,k} = \begin{pmatrix} \sum_{r=1}^{n} A_{1r}C_{r1} & \cdots & \sum_{r=1}^{n} A_{1r}C_{rp} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{mr}C_{r1} & \cdots & \sum_{r=1}^{n} A_{mr}C_{rp} \end{pmatrix}_{j,k}$$

$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

$$A_{j,\cdot} C_{\cdot,k} = (A_{j,1} \cdots A_{j,n}) \begin{pmatrix} C_{1,k} \\ \vdots \\ C_{n,k} \end{pmatrix} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

Therefore,

$$(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k}$$

3.50 Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \\ 31 \end{pmatrix}$$

A in Example 3.42 ; $C_{\cdot,2}$ in Example 3.42 ; $(AC)_{\cdot,2}$ in Example 3.42

Example 3.42

$$A = \boxed{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}} \qquad C = \begin{pmatrix} 6 & \boxed{5} & 4 & 3 \\ 2 & \boxed{1} & 0 & -1 \end{pmatrix}$$

$$AC = \begin{pmatrix} 10 & \boxed{7} & 4 & 1 \\ 26 & \boxed{19} & 12 & 5 \\ 42 & \boxed{31} & 20 & 9 \end{pmatrix}$$

$(AC)_{\cdot,2}$

$(AC)_{\cdot,2} = AC_{\cdot,2}$

3.52 Linear combination of columns

Suppose A is an $m \times n$ matrix and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an $n \times 1$ matrix

Then

$$Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$$

Proof:

$$c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n} = c_1 \begin{pmatrix} A_{1,1} \\ \vdots \\ A_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} A_{1,n} \\ \vdots \\ A_{m,n} \end{pmatrix} = \begin{pmatrix} c_1 A_{1,1} \\ \\ c_1 A_{m,1} \end{pmatrix} + \cdots + \begin{pmatrix} c_n A_{1,n} \\ \\ c_n A_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 A_{1,1} + \cdots + c_n A_{1,n} \\ \vdots \\ c_1 A_{m,1} + \cdots + c_n A_{m,n} \end{pmatrix}$$

$$AC = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} A_{1,1}c_1 + \cdots + A_{1,n}c_n \\ \\ A_{m,1}c_1 + \cdots + A_{m,n}c_n \end{pmatrix}$$

Therefore $AC = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$.

Similar exercise: Exercise 3.C.11 of Axler