

3.1 Matrices

3.30 Definition

Let m and n denote positive integers.

An $m \times n$ matrix A is a rectangular array of elements of \mathbb{F} , with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix}$$

jth row.

kth column

The notation $A_{j,k}$ denotes the entry in row j , column k of A .

row number column number

3.31 Example

$$\text{If } A = \begin{pmatrix} 8 & 4 & 5-3i \\ 1 & 9 & 7 \end{pmatrix}$$

$$\begin{aligned} \text{then } A_{1,1} &= 8 & A_{1,2} &= 4 & A_{1,3} &= 5-3i \\ A_{2,1} &= 1 & A_{2,2} &= 9 & A_{2,3} &= 7 \end{aligned}$$

3.32 Definition

Suppose $T \in L(V, W)$ and v_1, \dots, v_n is a basis of V and w_1, \dots, w_m is a basis of W .

The matrix of T with respect to these bases is the $m \times n$ matrix

$$M(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$$

whose entries $A_{j,k}$ are defined by $Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m$.

If the bases are understood, then we can just write $M(T)$ denote the matrix of T with respect to the bases.

$$M(T) = \begin{pmatrix} A_{1,1} & \dots & A_{1,k} & \dots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \dots & A_{j,k} & \dots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,k} & \dots & A_{m,n} \end{pmatrix}$$

k^{th} column of $M(T)$
 l^{th} column of $M(T)$

For any $k=1, \dots, n$, we have

$$T_{vk} = \begin{pmatrix} w_1 & \dots & w_m \end{pmatrix} \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

$1 \times m$ $m \times 1$

Identify all 1×1 matrices by their one entry

$$= \begin{pmatrix} A_{1,k}w_1 + \dots + A_{m,k}w_m \end{pmatrix}$$

1×1

$$= A_{1,k}w_1 + \dots + A_{m,k}w_m$$

$$= \sum_{j=1}^m A_{j,k}w_j$$

3.33 Example

Define $T \in L(\mathbb{F}^2, \mathbb{F}^3)$ by $T(x, y) = (x+3y, 2+5y, 7x+9y)$

Find the matrix of T with respect to the standard bases of $\mathbb{F}^2, \mathbb{F}^3$.

In other words, find

$$M(T) = M\left(T, \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right), \quad \begin{pmatrix} w_1 & w_2 & w_3 \\ (1,0,0), (0,1,0), (0,0,1) \end{pmatrix}$$

Standard basis of \mathbb{F}^2 Standard basis of \mathbb{F}^3

Solution:

we have

$$T(1,0) = (1, 2, 7)$$

$$T(0,1) = (3, 5, 9)$$

The matrix of T with respect to the standard bases is

$$M(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$

$$= (T(e_1), T(e_2))$$

$$T(1,0) = T_{e_1} = \sum_{j=1}^3 A_{j,1}w_j = A_{1,1}w_1 + A_{2,1}w_2 + A_{3,1}w_3$$

$$= 1(1,0,0) + 2(0,1,0) + 7(0,0,1) = (1, 2, 7)$$

$$T(0,1) = T_{e_2} = \sum_{j=1}^3 A_{j,2}w_j = A_{1,2}w_1 + A_{2,2}w_2 + A_{3,2}w_3$$

$$= 3(1,0,0) + 5(0,1,0) + 9(0,0,1) = (3, 5, 9)$$

3.34 Example

Let $1, x, x^2, \dots, x^m$ be the standard basis of $P_m(F)$.

Suppose $D \in L(P_3(\mathbb{R}), P_2(\mathbb{R}))$ is the differentiation map defined by

$$Dp = p'.$$

Find the matrix of D with respect to the standard basis of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$.

In other words, find

$$M(D) = M(D, \underbrace{(1, x, x^2, x^3)}_{\text{standard basis of } P_3(\mathbb{R})}, \underbrace{(1, x, x^2)}_{\text{standard basis of } P_2(\mathbb{R})})$$

Solution: Basis of $P_3(\mathbb{R})$ is $1, x, x^2, x^3$.

~~For all~~

Basis of $P_2(\mathbb{R})$ is $1, x, x^2$

For all positive integers n , $D(x^n) = (x^n)' = nx^{n-1}$

Input in $P_3(\mathbb{R})$	Corresponding vector in F^4
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$$1 \longleftrightarrow (1, 0, 0, 0)$$

$$x \longleftrightarrow (0, 1, 0, 0)$$

$$x^2 \longleftrightarrow (0, 0, 1, 0)$$

$$x^3 \longleftrightarrow (0, 0, 0, 1)$$

Output in $P_2(\mathbb{R})$	Corresponding vectors in F^3
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$$D(1) = 0 \longleftrightarrow (0, 0, 0)$$

$$D(x) = 1 \longleftrightarrow (1, 0, 0)$$

$$D(x^2) = 2x \longleftrightarrow (0, 2, 0)$$

$$D(x^3) = 3x^2 \longleftrightarrow (0, 0, 3)$$

$$M(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Input in $P_3(\mathbb{R})$

$$1 \quad M(D) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$x \quad M(D) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Output in $P_2(\mathbb{R})$

$$0$$

$$1$$

Verify with the differentiation map

$$D(1) = 0$$

$$D(x) = 1$$

$$x^2 \quad MD \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow x \quad D(x^2) = 2x$$

$$x^3 \quad MD \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow x^2 \quad D(x^3) = 3x^2$$

3.3 Continued

Addition and scalar multiplication of matrices

3.35 Definition

Matrix addition is the sum of two matrices of the same size, obtained adding the corresponding entries for the matrices.

$$\begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} \in \mathbb{F}^{m \times n} + \begin{pmatrix} C_{1,1} & \dots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \dots & C_{m,n} \end{pmatrix} \in \mathbb{F}^{m \times n} = \begin{pmatrix} A_{1,1} + C_{1,1} & \dots & A_{1,n} + C_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + C_{m,1} & \dots & A_{m,n} + C_{m,n} \end{pmatrix} \in \mathbb{F}^{m \times n}$$

In other words, $A_{j,k} + C_{j,k} = (A+C)_{j,k}$ for each $j=1, \dots, m$ and $k=1, \dots, n$

3.36 The matrix of the sum of linear maps

Suppose $S, T \in L(V, W)$ Then $M(S+T) = M(S) + M(T)$.

3.37 Definition

Scalar multiplication of a matrix is the product of a scalar and a matrix, obtained by multiplying each entry in the matrix by the scalar.

$$\lambda \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \dots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \dots & \lambda A_{m,n} \end{pmatrix}$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$ for each $j=1, \dots, m$ and $k=1, \dots, n$.

3.38 The matrix of a scalar times a linear map.

Suppose $\lambda \in \mathbb{F}$ and $T \in L(V, W)$. Then $M(\lambda T) = \lambda M(T)$.

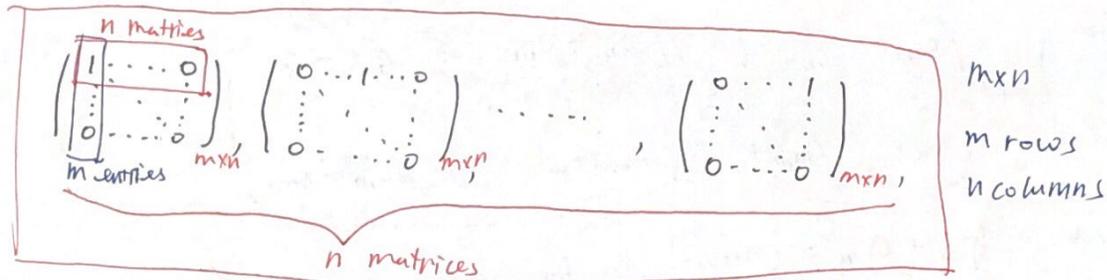
3.40 $\dim \mathbb{F}^{m \times n} = mn$

Let m and n be positive integers. Let $\mathbb{F}^{m \times n}$ is the set of all $m \times n$ matrices with entries in \mathbb{F} .

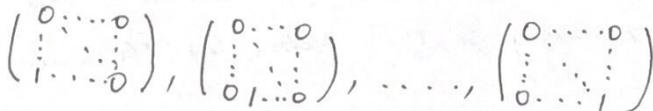
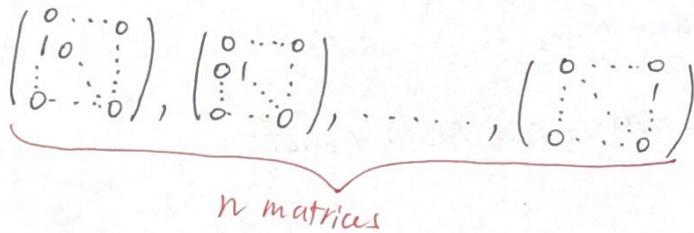
then $\dim \mathbb{F}^{m \times n} = mn$.

Proof: $\mathbb{F}^{m \times n}$ is a vector space with respect to the operations of matrix addition and scalar multiplication of a matrix.

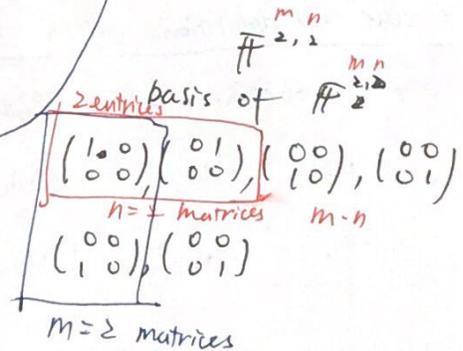
Now, $\dim \mathbb{F}^{m \times n} = mn$ because



m matrices



is a basis of $\mathbb{F}^{m \times n}$
(which you can verify)
So $\dim \mathbb{F}^{m \times n} = mn$.



$$\dim \mathbb{F}_{\mathbb{Z}}^{2,2} = mn = 2 \cdot 2 = 4.$$

Matrix Multiplication

Let v_1, \dots, v_n be a basis of V and w_1, \dots, w_m be a basis of W .

Also let U be a vector space and u_1, \dots, u_p is a basis of U .

Let $T: U \rightarrow V$ and $S: V \rightarrow W$ be linear maps.

We will show: $M(S \circ T) = M(S)M(T)$.

Suppose $M(S) = A$ and $M(T) = C$. For each $k=1, \dots, p$, we have

$$(S \circ T)u_k = S(Tu_k)$$

$$= S\left(\sum_{r=1}^n C_{rk} v_r\right) \quad \leftarrow \text{by Definition 3.32 of Axler}$$

$$= \sum_{r=1}^n C_{rk} S v_r \quad \leftarrow \text{since } S \text{ is linear}$$

$$= \sum_{r=1}^n C_{rk} \left(\sum_{j=1}^m A_{jr} w_j\right) \quad \leftarrow \text{by Definition 3.32 of Axler.}$$

$$= \sum_{j=1}^m \left(\sum_{r=1}^n A_{jr} C_{rk}\right) w_j$$

By Definition 3.32 of Axler, $D = M(S \circ T)$ is the matrix with entry in row j , column k , $D_{j,k} = \sum_{r=1}^n A_{jr} C_{rk}$.

3.41 Definition

Let A be an $m \times n$ matrix and C be an $n \times p$ matrix.

Then AC is an $m \times p$ matrix with entry in row j , column k :

$$(AC)_{j,k} = \sum_{r=1}^n A_{jr} C_{rk}$$

In other words,

$$AC = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \dots & A_{mn} \end{pmatrix} \begin{pmatrix} C_{11} & \dots & C_{1p} \\ \vdots & \ddots & \vdots \\ C_{n1} & \dots & C_{np} \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{r=1}^n A_{1r} C_{rk} & \dots & \sum_{r=1}^n A_{1r} C_{rp} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^n A_{mr} C_{r1} & \dots & \sum_{r=1}^n A_{mr} C_{rp} \end{pmatrix}$$

3.42 Example

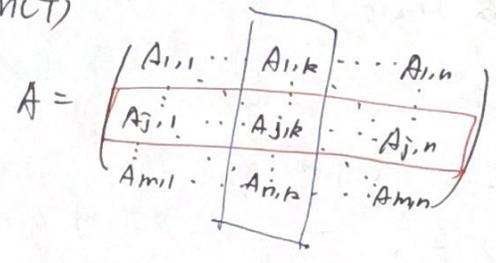
$$\begin{matrix}
 A & & C & & AC \\
 \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} & \begin{pmatrix} 6 & 3 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix} & = & \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 17 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix} \\
 3 \times 2 & 2 \times 4 & & & 3 \times 4
 \end{matrix}$$

3.43 The matrix of the product of linear maps

If $T \in L(U, V)$ and $S \in L(V, W)$, then

$$M(S \circ T) = M(S)M(T)$$

Let A be an $m \times n$ matrix,



Then

$$A_{j,\cdot} = (A_{j,1} \dots A_{j,n})$$

$$A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

3.45 Example

Let $A = \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix}$. Then $A_{2,\cdot} = (1 \ 9 \ 7)$ and $A_{\cdot 2} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

3.46 Example

Let $A = (3 \ 4)$ and $c = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$. Then $AC = (3 \ 4) \begin{pmatrix} 6 \\ 2 \end{pmatrix} = (26) = 26$

$(AC)_{1,1} = 36$ $(AC)_{1,\cdot} = 26$ $(AC)_{\cdot,1} = 26$

3.47 Entry of matrix product equals row times column

Suppose A is an $m \times n$ matrix and C is an $n \times p$ matrix. Then

$$(AC)_{j,k} = A_{j,\cdot} \cdot C_{\cdot,k}$$

Proof: $(AC)_{j,k} = \left(\begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \dots & C_{1,p} \\ \vdots & & \vdots \\ C_{n,1} & \dots & C_{n,p} \end{pmatrix} \right)_{j,k}$

$$= \left(\begin{pmatrix} \sum_{r=1}^n A_{j,r} C_{r,1} & \dots & \sum_{r=1}^n A_{j,r} C_{r,p} \\ \vdots & & \vdots \\ \sum_{r=1}^n A_{m,r} C_{r,1} & \dots & \sum_{r=1}^n A_{m,r} C_{r,p} \end{pmatrix} \right)_{j,k}$$

$$= \sum_{r=1}^n A_{j,r} C_{r,k}$$

$$A_{j,\cdot} \cdot C_{\cdot,k} = (A_{j,1} \dots A_{j,n}) \begin{pmatrix} C_{1,k} \\ \vdots \\ C_{n,k} \end{pmatrix} = \sum_{r=1}^n A_{j,r} C_{r,k}$$

Therefore, $(AC)_{j,k} = A_{j,\cdot} \cdot C_{\cdot,k}$

3.50 Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \\ 31 \end{pmatrix}$$

A $C_{\cdot,2}$ $(AC)_{\cdot,2}$
 in Example 3.42 in Example 3.42 in Example 3.42

Example 3.42

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \quad A$$

$$C = \begin{pmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & 1 \\ 10 & 7 & 4 & 1 \end{pmatrix} \quad C_{\cdot,2}$$

$$AC = \begin{pmatrix} 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix} \quad (AC)_{\cdot,2}$$

$$(AC)_{\cdot,2} = AC_{\cdot,2}$$

3.52 Linear combination of columns

Suppose A is an $m \times n$ matrix and $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$ is an $n \times 1$ matrix. Then $AC = C_1 A_{\cdot,1} + \dots + C_n A_{\cdot,n}$

Proof: $C_1 A_{:,1} + \dots + C_n A_{:,n} = C_1 \begin{pmatrix} A_{1,1} \\ \vdots \\ A_{m,1} \end{pmatrix} + \dots + C_n \begin{pmatrix} A_{1,n} \\ \vdots \\ A_{m,n} \end{pmatrix}$

$$= \begin{pmatrix} C_1 A_{1,1} \\ \vdots \\ C_1 A_{m,1} \end{pmatrix} + \dots + \begin{pmatrix} C_n A_{1,n} \\ \vdots \\ C_n A_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} C_1 A_{1,1} + \dots + C_n A_{1,n} \\ \vdots \\ C_1 A_{m,1} + \dots + C_n A_{m,n} \end{pmatrix}$$

$$AC = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1} C_1 + \dots + A_{1,n} C_n \\ \vdots \\ A_{m,1} C_1 + \dots + A_{m,n} C_n \end{pmatrix}$$

therefore, $AC = C_1 A_{:,1} + \dots + C_n A_{:,n}$

Similar exercise: ^{exercise} 3 c-11 of Axler