3.C Matrices

3.30 Definition

Let m and n denote positive integers.

An $m \times n$ matrix A is a rectangular array of elements of $\mathbb{F}$, with m rows and n columns:

$$A = \begin{array}{c} k^{\text{th}}\text{ column} \\ \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix} \end{array} \quad j^{\text{th}} \text{ row.}$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A.

row number   column number

3.31 Example

If $A = \begin{pmatrix} 8 & 4 & 5-3i \\ 1 & 9 & 7 \end{pmatrix}$

then $A_{1,1} = 8 \quad A_{1,2} = 4 \quad A_{1,3} = 5-3i$

$A_{2,1} = 1 \quad A_{2,2} = 9 \quad A_{2,3} = 7$

3.32 Definition

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W.

The <u>matrix of T</u> with respect to these bases is the $m \times n$ matrix

$$M(T, (v_1, \dots, v_n), (w_1, \dots, w_n))$$

whose entries $A_{j,k}$ are defined by $Tv_k = A_{1,k}w_1 + \dots + A_{m,k}w_m$.

If the bases are understood, then we can just write $M(T)$ denote the matrix of T with respect to the bases.

$$M(T) = \begin{array}{c} \\ \left(\begin{array}{ccccc} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{array}\right) \end{array}$$

k^{th} column of $M(T)$

For any $k = 1, \ldots, n$, we have
$$Tv_k = \underbrace{(w_1, \cdots w_m)}_{1 \times m} \underbrace{\begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}}_{m \times 1}$$
(kth column of $M(T)$)

Identify all 1×1 matrices by their one entry
$$= \underbrace{(A_{1,k} w_1 + \cdots + A_{m,k} w_m)}_{1 \times 1}$$
$$= A_{1,k} w_1 + \cdots + A_{m,k} w_m$$
$$= \sum_{j=1}^{m} A_{j,k} w_j$$

3.33 Example

Define $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ by $T(x,y) = (x+3y, 2x+5y, 7x+9y)$

Find the matrix of T with respect to the standard bases of $\mathbb{F}^2, \mathbb{F}^3$.

In other words, find
$$M(T) = M\Big(T, \underbrace{(\overset{v_1}{(1,0)}, \overset{v_2}{(0,1)})}_{\text{Standard basis of } \mathbb{F}^2}, \underbrace{(\overset{w_1}{(1,0,0)}, \overset{w_2}{(0,1,0)}, \overset{w_3}{(0,0,1)})}_{\text{Standard basis of } \mathbb{F}^3}\Big)$$

Solution: We have
$$T(1,0) = (1,2,7)$$
$$T(0,1) = (3,5,9)$$

The matrix of T with respect to the standard bases is
$$M(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$
$$\left(= (T(1,0), \ T(0,1)) \right)$$

$$T(1,0) = Tv_1 = \sum_{j=1}^{3} A_{j,1} w_j = A_{1,1} w_1 + A_{2,1} w_2 + A_{3,1} w_3$$
$$= 1(1,0,0) + 2(0,1,0) + 7(0,0,1) = (1,7,7)$$

$$T(0,1) = Tv_2 = \sum_{j=1}^{3} A_{j,2} w_j = A_{1,2} w_1 + A_{2,2} w_2 + A_{3,2} w_3$$
$$= 3(1,0,0) + 5(0,1,0) + 9(0,0,1) = (3,5,9)$$

3.34 Example

Let $1, x, x^2, \ldots, x^m$ be the standard basis of $P_m(\mathbb{F})$.

Suppose $D \in \mathcal{L}(P_3(\mathbb{R}), P_2(\mathbb{R}))$ is the differentiation map defined by

$$Dp = p'.$$

Find the matrix of D with respect to the standard basis of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$.

In other words, find

$$M(D) = M(D, \underbrace{\boxed{(1, x, x^2, x^3)}}_{\text{standard basis of } P_3(\mathbb{R})}, \underbrace{\boxed{(1, x, x^2)}}_{\text{standard basis of } P_3(\mathbb{R})})$$

Solution: Basis of $P_3(\mathbb{R})$ is $1, x, x^2, x^3$.

~~For all~~

Basis of $P_2(\mathbb{R})$ is $1, x, x^2$

For all positive integers n, $D(x^n) = (x^n)' = n x^{n-1}$

Input in $P_3(\mathbb{R})$		Corresponding vector in $\mathbb{F}^4$	Output in $P_2(\mathbb{R})$		Corresponding vectors in $\mathbb{F}^3$
1	$\longleftrightarrow$	$(1,0,0,0)$	$D(1) = 0$	$\longleftrightarrow$	$(0,0,0)$
x	$\longleftrightarrow$	$(0,1,0,0)$	$D(x) = 1$	$\longleftrightarrow$	$(1,0,0)$
x^2	$\longleftrightarrow$	$(0,0,1,0)$	$D(x^2) = 2x$	$\longleftrightarrow$	$(0,2,0)$
x^3	$\longleftrightarrow$	$(0,0,0,1)$	$D(x^3) = 3x^2$	$\longleftrightarrow$	$(0,0,3)$

$$M(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Input in $P_3(\mathbb{R})$		Output in $P_3(\mathbb{R})$	Verify with the Differentiation map
1	$M(D)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix}0\\0\\0\end{matrix}} & \begin{matrix}1 & 0 & 0\\ 0 & 2 & 0 \\ 0 & 0 & 3\end{matrix} \end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$	0	$D(1) = 0$
x	$M(D)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix} \begin{matrix}0\\0\\0\end{matrix} & \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \begin{matrix}0 & 0\\ 2 & 0 \\ 0 & 3\end{matrix} \end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$	1	$D(x) = 1$

x^2 $\quad \mathcal{M}D\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix} = 2\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad 2x \quad D(x^2)=2x$

x^3 $\quad \mathcal{M}D\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\3\end{pmatrix} = 3\begin{pmatrix}0\\0\\1\end{pmatrix}$ $\quad 3x^2 \quad D(x^3)= 3x^2.$

3C Continued

Addition and scalar multiplication of matrices

3.35 Definition

Matrix addition is the sum of two matrices of the same size, obtained adding the corresponding entries in the matrices.

$$\underbrace{\begin{pmatrix}A_{1,1} & \cdots & A_{1,n}\\ \vdots & \ddots & \vdots\\ A_{m,1} & \cdots & A_{m,n}\end{pmatrix}}_{\in \mathbb{F}^{m,n}} + \underbrace{\begin{pmatrix}C_{1,1} & \cdots & C_{1,n}\\ \vdots & & \vdots\\ C_{m,1} & \cdots & C_{m,n}\end{pmatrix}}_{\in \mathbb{F}^{n,n}} = \underbrace{\begin{pmatrix}A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n}\\ \vdots & \ddots & \vdots\\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n}\end{pmatrix}}_{\in \mathbb{F}^{m,n}}$$

In other words, $A_{j,k} + C_{j,k} = (A+C)_{j,k}$ for each $j=1,\dots,m$ and $k=1,\dots,n$

3.36 The matrix of the sum of linear maps

Suppose $S,T \in \mathcal{L}(V,W)$ Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

3.37 Definition

Scalar multiplication of a matrix is the product of a scalar and a matrix, obtained by multiplying each entry in the matrix by the scalar.

$$\lambda\begin{pmatrix}A_{1,1} & \cdots & A_{1,n}\\ \vdots & \ddots & \vdots\\ A_{m,1} & \cdots & A_{m,n}\end{pmatrix} = \begin{pmatrix}\lambda A_{1,1} & \cdots & \lambda A_{1,n}\\ \vdots & \ddots & \vdots\\ \lambda A_{m,1} & \cdots & \lambda A_{m,n}\end{pmatrix}$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$ for each $j=1,\dots,m$ and $k=1,\dots,m$.

3.38 The matrix of a scalar times a linear map.

Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

3.40 $\dim \mathbb{F}^{m,n} = mn$

Let m and n be positive integers. Let $\mathbb{F}^{m,n}$ is the set of all $m \times n$ matrices with entries in $\mathbb{F}$.

Then $\dim \mathbb{F}^{m,n} = mn$.

Proof: $\mathbb{F}^{m,n}$ is a vector space with respect to the operations of matrix addition and scalar multiplication of a matrix.

Now, $\dim \mathbb{F}^{m,n} = mn$ because

n matrices

$$\underbrace{\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{m\times n}, \begin{pmatrix} 0 & \cdots 1 \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{m\times n}, \cdots\cdots, \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{m\times n},}_{n \text{ matrices}}$$

m entries

$m \times n$

m rows

n columns

m matrices

$$\underbrace{\begin{pmatrix} 0 & \cdots & 0 \\ 1 & 0 & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 \\ 0 & 1 & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \cdots\cdots, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & 1 \\ 0 & \cdots & 0 \end{pmatrix}}_{n \text{ matrices}}$$

$$\vdots$$

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 1 \cdots & 0 \end{pmatrix}, \cdots\cdots, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

is a basis of $\mathbb{F}^{m,n}$ (which you can verify

So $\dim \mathbb{F}^{m,n} = mn$.

$\mathbb{F}^{\overset{m}{2},\overset{n}{2}}$

basis of $\mathbb{F}^{\overset{m}{2},\overset{n}{2}}$

2 entries

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{n=2 \text{ matrices}}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$m \cdot n$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$m = 2$ matrices

$$\dim \mathbb{F}^{\overset{m}{2},\overset{n}{2}} = mn = 2 \cdot 2 = 4.$$

Matrix Multiplication

Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_m$ be a basis of W.
Also let U be a vector space and $u_1, \ldots, u_p$ is a basis of U.
Let $T: U \to V$ and $S: V \to W$ be linear maps.
We will show: $M(ST) = M(S)M(T)$.

Suppose $M(S) = A$ and $M(T) = C$. For each $k = 1, \ldots p$, we have

$$
\begin{aligned}
(ST)u_k &= S(Tu_k) \\
&= S\left(\sum_{r=1}^{n} C_{r,k} v_r\right) && \text{by Definition 3.32 of Axler} \\
&= \sum_{r=1}^{n} C_{r,k} S v_r && \text{since } S \text{ is linear} \\
&= \sum_{r=1}^{n} C_{r,k} \left(\sum_{j=1}^{m} A_{j,r} w_j\right) && \text{by Definition 3.32 of Axler.} \\
&= \sum_{j=1}^{m} \left(\sum_{r=1}^{n} A_{j,r} C_{r,k}\right) w_j
\end{aligned}
$$

By Definition 3.32 of Axler, $D = M(ST)$ is the matrix with entry in row j, column k, $D_{j,k} = \sum_{r=1}^{n} A_{j,k} C_{r,k}$.

3.41 Definition

Let A be an $m \times n$ matrix and C be an $n \times p$ matrix.
Then AC is an $m \times p$ matrix with entry in row j, column k:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

In other words,

$$
AC = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix}
= \begin{pmatrix} \sum_{r=1}^{n} A_{1,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r} C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r} C_{r,p} \end{pmatrix}
$$

3.42 Example

$$\underset{3\times 2}{\overset{A}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}} \underset{2\times 4}{\overset{C}{\begin{pmatrix} 6 & 3 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix}}} = \underset{3\times 4}{\overset{AC}{\begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix}}}$$

3.43 The matrix of the product of linear maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then

$$\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$$

Let A be an $m \times n$ matrix, $A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix}$

Then $A_{j,\cdot} = (A_{j,1} \cdots A_{j,n})$

$$A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

3.45 Example

Let $A = \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix}$. Then $A_{2,\cdot} = (1\ 9\ 7)$ and $A_{\cdot,2} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

3.46 Example

Let $A = (3\ 4)$ and $C = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$. Then $AC = (3\ 4)\begin{pmatrix} 6 \\ 2 \end{pmatrix} = (26) = 26$

$(AC)_{1,1} = 36$ $\quad (AC)_{1,\cdot} = 26$ $\quad (AC)_{\cdot,1} = 26$

3.47 Entry of matrix product equals row times column

Suppose A is an $m \times n$ matrix and C is an $n \times p$ matrix. Then

$$(AC)_{j,k} = A_{j,\cdot}\, C_{\cdot,k}$$

Proof: $$(AC)_{j,k} = \left(\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} C_{1,1} & \cdots & C_{1,p} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,p} \end{pmatrix} \right)_{j,k}$$

$$= \left(\begin{pmatrix} \sum_{r=1}^{n} A_{1,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{1,r} C_{r,p} \\ \vdots & \ddots & \vdots \\ \sum_{r=1}^{n} A_{m,r} C_{r,1} & \cdots & \sum_{r=1}^{n} A_{m,r} C_{r,p} \end{pmatrix} \right)_{j,k}$$

$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

$$A_{j,\cdot}\, C_{\cdot,k} = \begin{pmatrix} A_{j,1} & \cdots & A_{j,n} \end{pmatrix} \begin{pmatrix} C_{1,k} \\ \vdots \\ C_{n,k} \end{pmatrix} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

Therefore, $(AC)_{j,k} = A_{j,\cdot}\, C_{\cdot,k}$

3.50 Example

$$\underset{\text{in Example 3.42}}{\underset{A}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}} \underset{\text{in Example 3.42}}{\underset{C_{\cdot,2}}{\begin{pmatrix} 5 \\ 1 \end{pmatrix}}} = \underset{\text{in Example 3.42}}{\underset{(AC)_{\cdot,2}}{\begin{pmatrix} 7 \\ 19 \\ 31 \end{pmatrix}}}$$

Example 3.42

$$A = \boxed{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}\ A$$

$$C = \begin{pmatrix} 6 & \boxed{\begin{matrix} 5 \\ 1 \end{matrix}} & 4 & 3 \\ 2 & & 0 & 1 \end{pmatrix}$$

$C_{\cdot,2}$

$$AC = \begin{pmatrix} 10 & \boxed{\begin{matrix} 7 \\ 19 \\ 31 \end{matrix}} & 4 & 1 \\ 26 & & 12 & 5 \\ 42 & & 20 & 9 \end{pmatrix}$$

$(AC)_{\cdot,2}$

$$(AC)_{\cdot,2} = A\,C_{\cdot,2}$$

3.52 Linear combination of columns

Suppose A is an $m \times n$ matrix and $C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an $n \times 1$ matrix.

Then $AC = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$

Proof: $c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n} = c_1 \begin{pmatrix} A_{1,1} \\ \vdots \\ A_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} A_{1,n} \\ \vdots \\ A_{m,n} \end{pmatrix}$

$$= \begin{pmatrix} c_1 A_{1,1} \\ \vdots \\ c_1 A_{m,1} \end{pmatrix} + \cdots + \begin{pmatrix} c_n A_{1,n} \\ \vdots \\ c_n A_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 A_{1,1} + \cdots + c_n A_{1,n} \\ \vdots \\ c_1 A_{m,1} + \cdots + c_n A_{m,n} \end{pmatrix}$$

$$AC = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1} c_1 + \cdots + A_{1,n} c_n \\ \vdots \\ A_{m,1} c_1 + \cdots + A_{m,n} c_n \end{pmatrix}$$

Therefore, $AC = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$

Similar excercise: 3 C.11 of Axler (Exercise)