3.D Invertibility and Isomorphic Vector Space

Invertible linear Maps

3.53 $\boxed{\text{Def}}$

- A linear map $T \in \mathcal{L}(V, W)$ is called Invertible if $\exists$ a linear map $S \in \mathcal{L}(W, V)$ such that $ST = I_V$ and $TS = I_W$, where I_V & I_W are identity maps on V & W, respectively.

Identity Maps

$I_V : V \to V$ — V, W vector spaces

$I_V(v) = v$ — v, w vectors in V, W

$I(w) : W \to W$

$I_w(w) = w$

$ST = I_V$	$TS = I_W$
$T: V \to W$	$S: W \to V$
$S: W \to V$	$T: V \to W$
$ST: V \to V$	$TS: W \to W$
$(I_V : V \to V)$	$(I_W : W \to W)$

If T is invertible with inverse S, the $ST = I_V$ & $TS = I_W$

3.54 Inverse is unique

An invertible linear map has a unique inverse.

proof: Suppose $T \in \mathcal{L}(V, W)$ is invertible, & let S and $\tilde{S}$ be inverse of T.

Then $S = SI_W = S(T\tilde{S}) = (ST)\tilde{S} = I_V \tilde{S} = \tilde{S}$

So $S = \tilde{S}$, means the inverse T is unique $\blacksquare$

Side Note: If T is invertible, then its inverse is denoted by T^{-1}.
If $T \in \mathcal{L}(V, W)$ is invertible, then $T^{-1} \in \mathcal{L}(W, V)$ is the unique element such that $T^{-1}T = I_V$ & $TT^{-1} = I_W$

7/16) 3.56 Invertibility is equivalent to surjectivity and injectivity

A linear map $T: V \to W$ invertible IFF T is injective & surjective.

Proof: * Forward direction: If T is invertible, then T is injective & surjective.

Suppose T is invertible. Then its inverse T^{-1} exists.

First, we will show that T is injective.

Suppose there exist $u, v \in V$ that satisfy $Tu = Tv$.

Then
$$\begin{aligned} u &= Iu \\ &= (T^{-1}T)u \\ &= T^{-1}(Tu) \\ &= T^{-1}(Tv) \\ &= (T^{-1}T)v \\ &= Iv = v \end{aligned}$$

So T is injective.

Next, we will show that T is surjective

Suppose we have an arbitrary vector $w \in W$ (we will argue $\forall w \in W$)

Then we have
$$\begin{aligned} w = Iw &= (TT^{-1})w \\ &= T(T^{-1}w) \end{aligned}$$

"v"

Since $w \in W$ & $T^{-1} \in \mathcal{L}(W, V)$, we have $T^{-1}w \in V$

So w is of form Tv for some $v \in V$, & so $w \in$ range T.

So $W \in$ range T. But 3.19, range T is a subspace of W.

So we have range $T = W$. So T is surjective.

* Backward: If T is injective & surjective, then T is invertible. Suppose T is injective and surjective. $\forall w \in W$, we can let $Sw \in V$ be a unique element that satisfies $T(Sw) = w$

or equivalently:

$$(T \circ S)w = w$$

So $T \circ S = I_W$, where I is the identity map on W.

- Next, we need to prove that $S \circ T = I_V$, where I_V is the identity map on V.

$$T((S \circ T)v) = (T \circ S \circ T)v$$
$$= (T \circ S)(Tv)$$
$$= I_W(Tv) = Tv.$$

Since T is injective, we get

$$(S \circ T)v = v$$

In other words, $S \circ T = I_V$

- Finally, show that $S: W \to V$ is linear (show $S \in \mathcal{L}(W, V)$)

Suppose we have $w_1, w_2 \in W$. Then, since T is linear, we have $T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2)$

$$= w_1 + w_2$$

Since Sw_1 & Sw_2 are unique elements of V that T maps to w_1 & w_2, respectively, it follows that $Sw_1 + Sw_2$ is a unique element of V that T maps to $w_1 + w_2$.

Furthermore, we have $S(w_1 + w_2) = S(T(Sw_1 + Sw_2))$

$$= (S \circ T)(Sw_1 + Sw_2)$$
$$= I_V(Sw_1 + Sw_2) = Sw_1 + Sw_2$$

satisfying additivity.

- Homogeneity: Suppose $w \in W$ and $\lambda \in \mathbb{F}$. Then since T is linear, $T(\lambda Sw) = \lambda T(Sw)$

$$= \lambda w$$

Since Sw is the unique element of V that T maps to w, it follows that λSw is the unique element of V that T maps to λw.

Furthermore, we have $S(\lambda w) = S(T(\lambda Sw)) = (S \circ T)(\lambda Sw)$
$= I_V(\lambda Sw) = \lambda Sw$, satisfying homogeneity.

Isomorphic Vector Spaces

3.58 Def)

- An isomorphism is an invertible linear map.
- Two vector spaces V & W are called isomorphic if there exists an isomorphism $T: V \to W$

3.59 Dimension shows whether vector spaces are isomorphic

Let V & W be finite-dim vector spaces over $\mathbb{F}$

Then V & W are isomorphic IFF $\dim V = \dim W$

- Proof direction: If V & W are isomorphic, then $\dim V = \dim W$. Suppose V & W are isomorphic, $\exists$ an isomorphism $T: V \to W$. Since T is isomorphism, it is invertible. By 3.56, T is injective & surjective. In other words, we have $\text{null}\, T = \{0\}$ (3.16) and range $T = W$. By the fundamental theorem of linear maps (3.22), we obtain $\dim V = \dim \text{null}\, T + \dim \text{range}\, T = \dim \{0\} + \dim W = 0 + \dim W = \dim W$.
- Backward direction:

  If $\dim V = \dim W$, then V & W are isomorphic.

  Since V and W are finite-dim, by 2.32, $\exists$ a basis $v_1, \dots, v_n$ of V and $w_1, \dots, w_n$ of W, $n = \dim V = \dim W$

Define $T: V \to W$ by

$$T(c_1v_1 + \dots + c_nv_n) = c_1w_1 + \dots + c_nw_n.$$

Then T is linear and well-defined, according to the proof of 3.5. Since $w_1, \dots, w_n$ is a basis of W, it spans W. Since every vector in W can be written uniquely as $c_1w_1 + \dots + c_nw_n$, T is surjective.

Suppose $c_1v_1 + \dots + c_nv_n \in \text{null}\, T$. Then $T(c_1v_1 + \dots + c_nv_n) = 0$, or $c_1w_1 + \dots + c_nw_n = 0$

Since $w_1, \dots, w_n$ is a basis of W, it is LI.

In other words, if $c_1, \dots, c_n \in \mathbb{F}$ satisfy $c_1w_1 + \dots + c_nw_n = 0$ then $c_1 = 0, \dots, c_n = 0$

So, $c_1v_1 + \dots + c_nv_n = 0v_1 + \dots + 0v_n = 0$

Therefore, null $T \subset \{0\}$. Also, since $T(0) = 0$, we have $\{0\} \subset$ null T. Therefore, null $T = \{0\}$. By 3.16, T is injective.
Finally as T is both injective & surjective.
By 3.56 T is an isomorphism.

• Note: if $n = \dim V$, then

$$\dim V = n = \dim \mathbb{F}^n.$$

And by 3.59, V is isomorphic to $\mathbb{F}^n$.

3.60 $\mathcal{L}(V, W)$ & $\mathbb{F}^{m,n}$ are isomorphic

Then $M : \mathcal{L}(V, W) \to \mathbb{F}^{m,n}$ is an isomorphism

Proof: From section 3C, $M : \mathcal{L}(V, W) \to \mathbb{F}^{m,n}$ is linear.
We will prove that M is injective and surjective.
First, we will show that M is injective.
Suppose $T \in$ null M. Then $M(T) = 0$ so we have:

$$\begin{vmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \ddots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \ddots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{vmatrix} = \begin{vmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{vmatrix}$$

In other words, $A_{j,k} = 0$ $\exists$ $j = 1, \dots, m$ and $k = 1, \dots, n$.
Therefore, $\exists$ $k = 1, \dots, n$

$$T v_k = \sum_{j=1}^{m} A_{j,k} w_j \quad \text{by 3.32}$$
$$= \sum_{j=1}^{n} 0 w_j = 0$$

Since $v_1, \dots, v_n$ is a basis of V, we conclude $T = 0$.
So null $M \subset \{0\}$, or null $M = \{0\}$. By 3.16, M is injective.

Next prove M is surjective. Suppose $A \in \mathbb{F}^{m,n}$,
Define $T \in \mathcal{L}(V, W)$ by $T v_k = \sum_{j=1}^{m} A_{j,k} w_j$
$\exists$ $k = 1, \dots, n$ then

$$M(T) = \begin{vmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{vmatrix} = A$$

So $A \in$ range M, and so $\mathbb{F}^{m,n} \subset$ range M.
But 3.19, range M is a subspace of $\mathbb{F}^{m,n}$. So range $M = \mathbb{F}^{m,n}$, and so M is surjective.
Therefore $M : \mathcal{L}(V, W) \to \mathbb{F}^{m,n}$ is both injective & surjective.

By 3.56, M is invertible.
Since M is both linear & invertible, it is isomorphism.

3.61 $\dim \mathcal{L}(V,W) = \dim V \cdot \dim W$

Suppose V & W are finite-dim. vectors spaces. Then $\mathcal{L}(V,W)$, is finite-dim & $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$.

· Proof: Suppose $m = \dim V$ & $n = \dim W$
By 3.60 $\mathcal{L}(V,W)$ & $\mathbb{F}^{m,n}$ are isomorphic
By 3.59, $\dim \mathcal{L}(V,W) = \dim \mathbb{F}^{m,n}$
By 3.40, $\dim \mathbb{F}^{m,n} = mn$. So $\dim \mathcal{L}(V,W) = \dim \mathbb{F}^{m,n}$
$= mn = \dim V \dim W$

linear maps thought of as Matrix Multiplication.

3.62 $\boxed{\text{def}}$
Suppose $v \in V$ & $v_1, \dots, v_n$ is a basis of V
Then the matrix of a vector v with respect to this bases is the $n \times 1$ matrix $M(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$,

where $c_1, \dots, c_n \in \mathbb{F}$ satisfy
$v = c_1 v_1 + \dots + c_n v_n$

3.63 Example $2 - 7x + 0x^2 + 5x^3$

· The matrix of $2 - 7x + 5x^3$ to standard basis $1, x, x^2, x^3$ of $P_3(\mathbb{R})$ is
$M(2 - 7x + 5x^3) = \begin{pmatrix} 2 \\ -7 \\ 0 \\ 5 \end{pmatrix}$

· Matrix of $x \in \mathbb{F}^n$ with respect to the standard basis of $\mathbb{F}^n$ is $M(x) = M(x_1, \dots, x_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

3.64 $M(T)_{\cdot,k} = M(Tv_k)$
Suppose $T \in \mathcal{L}(V,W)$ & $v_1, \ldots, v_n$ is a basis of V & $w_1, \ldots, w_m$ is a basis of W. Let $k = 1, \ldots, n$.
Then the kth column of $M(T)$ equals $M(Tv_k)$:
$$(M(T))_{\cdot,k} = M(Tv_k).$$
Proof: let $A = M(T)$. Then we have

$$M(T)_{\cdot,k} = A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix} \text{ and } M(Tv_k) = M(A_{1,k}w_1 + \cdots + A_{m,k}w_m)$$

by 3.32 $= \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$

Therefore, $M(T)_{\cdot,k} = M(Tv_k)$ ∎

3.65 <u>linear maps act like matrices multiplication</u>

Suppose $T \in \mathcal{L}(V,W)$ & $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W.
Then $M(Tv) = M(T)M(v)$
Proof: Since $v_1, \ldots, v_n$ is a basis of V, we can write every $v \in V$ uniquely as
$$v = c_1v_1 + \cdots + c_nv_n$$
for some $c_1, \ldots, c_n \in \mathbb{F}$. Then we have $Tv = T(c_1v_1 + \cdots + c_nv_n)$
$= T(c_1v_1) + \cdots + T(c_nv_n) = c_1Tv_1 + \cdots + c_nTv_n$
So $M(Tv) = M(c_1Tv_1 + \cdots + c_nTv_n) = M(c_1Tv_1) + \cdots + M(c_nTv_n)$
$= c_1M(Tv_1) + \cdots + c_nM(Tv_n) = c_1(M(T))_{\cdot,1} + \cdots + c_n(M(T))_{\cdot,n}$
by 3.64
$= M(T)M(v)$ by 3.62

Operations
~~3.67 def~~
- linear map $T: V \to V$ is called an operator.
- $\mathcal{L}(V)$ denotes the set of all operators on V: $\mathcal{L}(V) = \mathcal{L}(V,V)$

3.69 Injectivity is equivalent to surjectivity in finite dim.
Suppose V is a finite-dim vector space & $T \in \mathcal{L}(V)$
then the following are equivalent:

a) T is invertible;
b) T is injective;
c) T is surjective.

Proof: • a) implies b):
Suppose (a) holds; suppose T is invertible. By 3.56, T is injective, which is b).
• b) implies c):
Suppose b) holds; suppose T is injective. By 3.16, null $T = \{0\}$ By fundamental theorem of linear maps, 3.22, we have dim range T = dim V − dim null T
$= \dim V - \dim\{0\} = \dim V - 0 = \dim V.$
By Exercise 2.C.1, we have range $T = V$ so T is surjective; which is c).
• c) implies a):
Suppos c) holds; suppose T is surjective. then range $T = V$.
By fundamental theorem of linear Maps (3.22) we have dim null T = dim V − dim range T
$= \dim V - \dim V = 0$
$= \dim\{0\}$
By exercise 2.C.1, we have null $T = \{0\}$. By 3.16, T is injective. So T is both injective & surjective, which is a).

3.E Products and Quotients of Vector Spaces

3.71 Def)

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$

- The product $V_1 \times \dots \times V_m$ is defined by
  $V_1 \times \dots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}$
- Addition on $V_1 \times \dots \times V_m$ is defined by $(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m)$
- Scalar multiplication on $V_1 \times \dots \times V_m$ is defined by $\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m)$

3.72 Example: $\underbrace{(5-6x+4x^2, (3,8,7))}_{\text{length } 2} \in P_2(\mathbb{R}) \times \mathbb{R}^3$

3.73 Product of vector spaces is a vector space

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$, then $V_1 \times \dots \times V_m$ is a vector space over $\mathbb{F}$.

Proof: Let $u_i, v_i, w_i \in V_i$ $\exists$ $i = 1, \dots, m$, & let $\lambda \in \mathbb{F}$

- Commutativity: Since each V_i is a vector space, we have $u_i + v_i = v_i + u_i$.

So we have: $(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m)$
$= (v_1 + u_1, \dots, v_m + u_m)$
$= (v_1, \dots, v_m) + (u_1, \dots, u_m)$

- Associativity: Since V_i is a vector space, we have $(u_i + v_i) + w_i = u_i + (v_i + w_i)$

So we have $((u_1, \dots, u_m) + (v_1, \dots, v_m)) + (w_1, \dots, w_m)$
$= ((u_1 + v_1, \dots, u_m + v_m) + w_m) = (u_1 + (v_1 + w_1), \dots, u_m + (v_m + w_m))$
$= (u_1, \dots, u_m) + (v_1 + w_1, \dots, v_m + w_m) = (u_1, \dots, u_m) + ((v_1, \dots, v_m) + (w_1, \dots, w_m))$.

- Additive identity: We have $(0, \dots, 0) \in V_1 \times \dots \times V_m$. And it satisfies ~~[illegible]~~ $(v_1, \dots, v_m) + (0, \dots, 0) = (v_1 + 0, \dots, v_m + 0)$
$= (v_1, \dots, v_m)$

So $(0, \dots, 0)$ is the additive identity of $V_1 \times \dots \times V_m$.

· Additive Inverse: We have $(-v_1, \ldots, -v_m) \in V_1, \ldots, V_m$.
And it satisfies $(v_1, \ldots, v_m) + (-v_1, \ldots, -v_m)$
$= (v_1 + (-v_1), \ldots, v_m + (-v_m)) = (v_1 - v_1, \ldots, v_m - v_m)$
$= (0, \ldots, 0)$
So $(-v_1, \ldots, -v_m)$ is the additive inverse of $V_1 \times \ldots \times V_m$

Multiplicative Identity: We have $1(v_1, \ldots, v_m) =$
$(1v_1, \ldots, 1v_m) = v_1, \ldots, v_m)$
Distributive properties: $\forall a, b \in \mathbb{F}$, we have
$a((u_1, \ldots, u_m) + (v_1, \ldots, v_m)) = a(u_1 + v_1, \ldots, u_m + v_m)$
$= (a(u_1 + v_1), \ldots, a(u_m + v_m))$
$= (au_1 + av_1, \ldots, au_m + av_m)$
$= (au_1, \ldots, au_m) + (av_1, \ldots, av_m)$
$= a(u_1, \ldots, u_m) + a(v_1, \ldots, v_m)$
and $(a+b)(v_1, \ldots, v_m) = ((a+b)v_1, \ldots, (a+b)v_m)$
$(av_1 + bv_1, \ldots, av_m + bv_m) = (av_1, \ldots, av_m) + (bv_1, \ldots, bv_m)$
$= a(v_1, \ldots, v_m) + b(v_1, \ldots, v_m)$

3.72 Example
Show that $\mathbb{R}^2 \times \mathbb{R}^3$ is isomorphic to $\mathbb{R}^5$
Note that, as vector spaces, $\mathbb{R}^2 \times \mathbb{R}^3 \neq \mathbb{R}^5$ because elements $((x_1, x_2), (x_3, x_4, x_5))$ of $\mathbb{R}^2 \times \mathbb{R}^3$ have length 2 but elements $(x_1, x_2, x_3, x_4, x_5)$ of $\mathbb{R}^5$ have length 5.
Proof:
Define $T: \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$ by
$T((x_1, x_2), (x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4, x_5)$
First show T is injective.
let $((x_1, x_2), (x_3, x_4, x_5)) \in \operatorname{null} T$, which means
$T((x_1, x_2), (x_3, x_4, x_5)) = (0, 0, 0, 0, 0)$
Then we have $(0, 0, 0, 0, 0) = T((x_1, x_2), (x_3, x_4, x_5))$
$= (x_1, x_2, x_3, x_4, x_5)$

So $x_1 = 0 = x_2 = x_3 = x_4 = x_5$

This means:

$((x_1, x_2),(x_3, x_4, x_5)) = ((0,0),(0,0,0))$

So null $T \in \{(0,0),(0,0,0)\}$

$T(0,0),(0,0,0) = (0,0,0,0,0)$

we also have $\{(0,0),(0,0,0)\} \in$ null T

Therefore, null $T = \{(0,0),(0,0,0)\}$

By 3.16, T is injective.

Next, show that T is surjective.

$\forall (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, we have

$(x_1, x_2, x_3, x_4, x_5) = T((x_1, x_2),(x_3, x_4, x_5)) \in$ range T

So $\mathbb{R}^5 \in$ range T. But range T is a subspace of $\mathbb{R}^5$.

So we have range $T = \mathbb{R}^5$

So T is surjective.

So, by 3.56, T is invertible.

Next, we will show that T is linear.

- Additivity $\forall x_1, \dots, x_5, y_1, \dots, y_5 \in \mathbb{R}$ we have:

$T(((x_1, x_2),(x_3, x_4, x_5)) + ((y_1, y_2),(y_3, y_4, y_5)))$
$= T((x_1 + y_1, x_2 + y_2),(x_3 + y_3, x_4 + y_4, x_5 + y_5))$
$= (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + y_5)$
$= (x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)$
$= T((x_1, x_2),(x_3, x_4, x_5)) + T((y_1, y_2),(y_3, y_4, y_5))$

- Homogeneity: $\forall \lambda \in \mathbb{F}$ and $\forall x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$

$T(\lambda((x_1, x_2),(x_3, x_4, x_5))) = T((\lambda x_1, \lambda x_2),(\lambda x_3, \lambda x_4, \lambda x_5))$
$= (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) = \lambda(x_1, \dots, x_5)$
$= \lambda T((x_1, x_2),(x_3, x_4, x_5))$

So, T is linear.

So T is invertible & linear.

Therefore, T is an isomorphism ∎

3.75 Example

Find a basis of $P_2(\mathbb{R}) \times \mathbb{R}^2$

Solution: $(1,(0,0)), (x,(0,0)), (x^2,(0,0)), (0,(1,0)), (0,(0,1))$

$1, x, x^2$ is a basis of $P_2(\mathbb{R})$

3.76 Dim. of product is the sum of dims.

Suppose $V_1, \dots, V_m$ are finite-dim vector spaces.
Then $V_1 \times \dots \times V_m$ is finite dim and $\dim(V_1 \times \dots \times V_m)$
$= \dim V_1 + \dots + \dim V_m$

Proof:
Choose a basis of each V_i. $\exists$ basis vector of each V_i
consider the element of $V_1 \times \dots \times V_m$ that equals
the basis vector in the ith slot & 0 in the other
slots. The list of all such vectors is linearly
independent and spans $V_1 \times \dots \times V_m$, so it is a basis
of $V_1 \times \dots \times V_m$, with length: $\dim V_1 + \dots + \dim V_m$ ∎

Products & Direct Sums

3.77 Products & direct sums

Suppose that $U_1, \dots, U_m$ are subspaces of V.
Define a linear map
$\Gamma : U_1 \times \dots \times U_m \to U_1 + \dots + U_m$ by
$\Gamma(u_1, \dots, u_m) = u_1 + \dots + u_m$

The $U_1 + \dots + U_m$ is a direct sum IFF Γ is
injective.

Proof: Forward: If $U_1 + \dots + U_m$ is a direct sum,
the Γ is injective. suppose $(u_1, \dots, u_m) \in \operatorname{null} \Gamma$,
so that $\Gamma(u_1, \dots, u_m) = \underbrace{0 + \dots + 0}_{m}$
Since $U_1 + \dots + U_m$ is a direct sum, by 1.44, the only
way to write the zero vector $0 + \dots + 0$ is to
take $u_1 = 0, \dots, u_m = 0$.
So $(u_1, \dots, u_m) = 0$ and so $\operatorname{null} \Gamma = \{0\}$. By 3.16, Γ is injective

Backward: If Γ is injective, then $V_1 + \dots + V_m$ is a direct sum. Since Γ is injective, by 3.16, we have null $\Gamma = \{(0, \dots, 0)\}$

So the only way to write 0 as 0 is to take $u_1 = 0, \dots, u_m = 0$. By 1.44, $V_1 + \dots + V_m$ is a direct sum. ☒

3.78 A sum is a direct sum IFF dims. add up.

Suppose V is finite-dim and $V_1, \dots, V_m$ are subspaces of V. Then $V_1 + \dots + V_m$ is a direct sum IFF

$$\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m$$

Proof: By the proof of 3.77, the map $\Gamma: V_1 \times \dots \times V_m \to V_1 + \dots + V_m$ defined by $\Gamma(u_1, \dots, u_m) = u_1 + \dots + u_m$ is surjective. So the fundamental theorem of linear maps (3.22) gives us $\dim(V_1 + \dots + V_m) = \dim \text{range}\, \Gamma$.

$$= \dim(V_1 \times \dots \times V_m) - \dim \text{null}(\Gamma)$$
$$= \dim(V_1 \times \dots \times V_m) - \dim\{0\}$$
$$= \dim(V_1 \times \dots \times V_m) \text{ IFF } \Gamma \text{ is injective.}$$

Combine 3.77 & 3.76 to conclude, $V_1 + \dots + V_m$ is a direct sum IFF, $\dim(V_1 + \dots + V_m) = \dim(V_1 \times \dots \times V_m)$

$$= \dim V_1 + \dots + \dim V_m. \text{ by 3.76} ☒$$