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3D

3.56 Invertibility is equivalent to surjectivity and injectivity

A linear map $T: V \to W$ invertible iff T is injective and surjective.

Proof:

Let $T: V \to W$ be a linear map.

Forward direction: If T is invertible, then T is injective & surjective.

Suppose T is invertible. Then it's inverse T^{-1} exists.

First, we will show that T is injective.

Suppose there exist $u, v \in V$ that satisfy $Tu = Tv$.

Then
$$\begin{aligned} u &= Iu \\ &= (T^{-1}I)u \\ &= T^{-1}(Tu) \\ &= T^{-1}(Tv) \\ &= (T^{-1}T)v \\ &= Iv \\ &= v \end{aligned}$$

$\therefore T$ is injective.

Next, we will show that T is surjective.

Suppose we have an arbitrary vector $w \in W$ (We will argue for all $w \in W$)

Then we have:

$$w = Iw$$

$= (TT^{-1})w$

$= T(T^{-1}w)$

Since $w \in W$ and $T^{-1} \in \mathcal{L}(W,V)$, we have $T^{-1}w \in V$.

So w is of the form Tv for some $v \in V$, and so $w \in \text{range } T$.

So $W \subset \text{range } T$. But 3.19 Axler, range T is a subspace of W.

So we have range $T = W$, so T is surjective.

Backward direction:

If T is injective and surjective, then T is invertible.

Suppose T is injective and surjective. For each $w \in W$, we can let $Sw \in V$ be a unique element that satisfies

because T injective $T(Sw) = w$

Because T is surjective, $w \in \text{range } T$

Or equivalently,

$$(T \circ S)w = w$$

So $T \circ S = I_W$, where I is the identity map on W.

Next, we need to prove that $S \circ T = I_V$, where I_V is the identity map on V.

We have:

$$T((S \circ T)v) = (T \circ S \circ T)v$$
$$= (T \circ S)(Tv)$$

Since T is injective, we get

$$(S \circ T)v = v$$

In other words,

$$S \circ T = I_V$$

Finally, we will show that $S: W \to V$ is linear (show $S \in \mathcal{L}(W,V)$)

- Suppose we have $w_1, w_2 \in W$. Then, since T is linear, we have

$$
\begin{aligned}
T(Sw_1 + Sw_2) &= T(Sw_1) + T(Sw_2) \\
&= w_1 + w_2
\end{aligned}
$$

Since Sw_1 and Sw_2 are unique elements of V that T maps to w_1 and w_2, respectively, it follows that $Sw_1 + Sw_2$ is a unique element of V that T maps to $w_1 + w_2$.

Furthermore, we have

$$
\begin{aligned}
S(w_1 + w_2) &= S(T(Sw_1 + Sw_2)) \\
&= (S \circ T)(Sw_1 + Sw_2) \\
&= I_V(Sw_1 + Sw_2) \\
&= Sw_1 + Sw_2
\end{aligned}
$$

· Satisfying additivity.

- Homogeneity:

Suppose we have $w \in W$ and $\lambda \in \mathbb{F}$. Then, since T is linear. we have

$$
\begin{aligned}
T(\lambda Sw) &= \alpha T(Sw) \\
&= \lambda w
\end{aligned}
$$

Since Sw is the unique element of V that T maps to w, it follows that λSw is the unique element of V that T maps to λw.

Furthermore, we have

$$
\begin{aligned}
S(\lambda w) &= S(T(\lambda Sw)) \\
&= (S \circ T)(\lambda Sw) \\
&= I_V(\lambda Sw) \\
&= \lambda Sw
\end{aligned}
$$

satisfying homogeneity

$\therefore S \in \mathcal{L}(W, V)$ $\therefore$ T is invertible.

Isomorphic Vector Spaces

3.58 Definition

- An isomorphism is an invertible linear map.
- Two vector spaces V and W are called isomorphic if there exists an isomorphism $T: V \to W$.

3.59 Dimension shows whether vector spaces are isomorphic.

Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Then V and W are isomorphic iff $\dim V = \dim W$.

Proof:

$\Rightarrow$ Forward direction:

If V and W are isomorphic, then $\dim V = \dim W$.

Suppose V and W are isomorphic, there exists an isomorphism $T: V \to W$. Since T is isomorphism, it is invertible. By 3.56, T is injective and surjective. In other words, we have $\operatorname{null} T = \{0\}$ and $\operatorname{range} T = W$. By the Fundamental Theorem of Linear Maps, we obtain:

$$
\begin{aligned}
\dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\
&= \dim \{0\} + \dim W \\
&= 0 + \dim W \\
&= \dim W.
\end{aligned}
$$

$\Leftarrow$ Backward direction:

If $\dim V = \dim W$, then V and W are isomorphic.

Since V and W are finite-dimensional, by 2.32, there exist a basis $v_1, \dots, v_n$ of V and $w_1, \dots, w_n$ of W, where $n = \dim V = \dim W$

Define $T: V \to W$ by

$$T(c_1 v_1 + \dots + c_n v_n) = c_1 w_1 + \dots + c_n w_n$$

Then T is linear and well-defined, according to the proof of 3.5

Since $w_1, \dots, w_n$ is a basis of W, it spans W. Since every vector in W can be written uniquely as $c_1 w_1 + \dots + c_n w_n$, T is surjective.

Since $w_1, \dots, w_n$ is a basis of W, it is linearly independent.

In other words, if $c_1, \dots, c_n \in \mathbb{F}$ satisfy

$$c_1 w_1 + \dots + c_n w_n = 0$$

then

$$c_1 = 0, \dots, c_n = 0$$

Suppose $c_1 v_1 + \dots + c_n v_n \in \operatorname{null} T$. Then

$$T(c_1 v_1 + \dots + c_n v_n) = 0,$$

or

$$c_1 w_1 + \dots + c_n w_n = 0$$

Consequently,

$$\begin{aligned} c_1 v_1 + \dots + c_n v_n &= 0 v_1 + \dots + 0 v_n \\ &= 0 \end{aligned}$$

$\therefore$ $\operatorname{null} T \subset \{0\}$. Also, since $T(0) = 0$, we have $\{0\} \subset \operatorname{null} T$.

$\therefore$ $\operatorname{null} T = \{0\}$. By 3.16 Axler, T is injective.

Finally, as T is both injective and surjective.

By 3.56, T is an isomorphism.

Note: if $n = \dim V$, then

$$\dim V = n = \dim \mathbb{F}^n$$

And 3.59 says that V is isomorphic to $\mathbb{F}^n$.

3.60 $\mathcal{L}(V,W)$ and $\mathbb{F}^{m,n}$ are isomorphic

Suppose $v_1,\dots,v_n$ is a basis of V and $w_1,\dots,w_n$ is a basis of W.

Then $\mathcal{M}: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is an isomorphism.

Proof:

From 3C, $\mathcal{M}: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is linear. We will prove that $\mathcal{M}$ is injective and surjective.

First, we will show that $\mathcal{M}$ is injective.

Suppose $T \in \operatorname{null} \mathcal{M}$. Then $\mathcal{M}(T)=0$. So we have:

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

In other words, $A_{j,k}=0$ for each $j=1,\dots,m$ and $k=1,\dots,n$

$\therefore$ for each $k=1,\dots,n$

$$\begin{aligned} Tv_k &= \sum_{j=1}^{m} A_{j,k} w_j \quad \text{be definition 3.32} \\ &= \sum_{j=1}^{n} 0 w_j \\ &= 0 \end{aligned}$$

Next, we will prove that $\mathcal{M}$ is surjective. Suppose $A \in \mathbb{F}^{m,n}$, which means A is an $m \times n$ matrix. Define $T \in \mathcal{L}(V,W)$ by

$$Tv_k = \sum_{j=1}^{m} A_{j,k} W_j$$

for each $k=1,\dots,n$ Then

$$\mathcal{M}(T) = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} = A$$

So $A \in \operatorname{range} \mathcal{M}$, and so $\mathbb{F}^{m,n} \subset \operatorname{range} \mathcal{M}$.

By 3.19, range $\mathcal{M}$ is a subspace of $\mathbb{F}^{m,n}$.

So range $\mathcal{M} = \mathbb{F}^{m,n}$, and so $\mathcal{M}$ is surjective.

Therefore, $\mathcal{M}: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is both injective and surjective

By 3.56, $\mathcal{M}$ is invertible.

Since $\mathcal{M}$ is both linear and invertible, it is an isomorphism.

3.61 $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$

Suppose V and W are finite-dimensional vector spaces. Then $\mathcal{L}(V,W)$ is finite-dimensional and.

$$\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$$

Proof: Suppose $m = \dim V$ and $n = \dim W$

By 3.60, $\mathcal{L}(V,W)$ and $\mathbb{F}^{m,n}$ are isomorphic

3.59, $\dim \mathcal{L}(V,W) = \dim \mathbb{F}^{m,n}$

3.40, $\dim \mathbb{F}^{m,n} = mn$

$$\therefore \dim \mathcal{L}(V,W) = \dim \mathbb{F}^{m,n} = mn = (\dim V)(\dim W)$$

Linear maps thought of as Matrix multiplication.

3.62 Definition

Suppose $v \in V$ and $v_1, \dots, v_n$ is a basis of V.

Then the matrix of a vector v with respect to this basis is the $n \times 1$ matrix

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where $c_1,\dots,c_n\in\mathbb{F}$ satisfy

$$v=c_1v_1+\dots+c_nv_n$$

3.63 Example

- The matrix of $2-7x+5x^3$ with respect to the standard basis $1, x, x^2, x^3$ of $\mathcal{P}_3(\mathbb{R})$ is

$$\mathcal{M}(2-7x+5x^3)=\begin{pmatrix}2\\-7\\0\\5\end{pmatrix}$$

- The matrix of $x\in\mathbb{F}^n$ with respect to the standard basis of $\mathbb{F}^n$ is

$$\mathcal{M}(x)=\mathcal{M}((x_1,\dots,x_n))=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$$

3.64 $\mathcal{M}(T)_{\cdot,k}=\mathcal{M}(Tv_k)$

Suppose $T\in\mathcal{L}(V,W)$ and $v_1,\dots,v_n$ is a basis of V and $w_1,\dots,w_m$ is a basis of W, let $k=1,\dots,n$. Then the k^{th} column of $\mathcal{M}(T)$ equals $\mathcal{M}(Tv_k)$. In other words,

$$(\mathcal{M}(T))_{\cdot,k}=\mathcal{M}(Tv_k)$$

Proof:

Let $A=\mathcal{M}(T)$. Then we have

$$\mathcal{M}(T)_{\cdot,k}=A_{\cdot,k}=\begin{pmatrix}A_{1,k}\\ \vdots\\ A_{m,k}\end{pmatrix}$$

and $\mathcal{M}(Tv_k)=\mathcal{M}(\overset{c_1}{A_{1,k}}w_1+\dots+\overset{c_n}{A_{m,k}}w_m)$ by 3.32

$$=\begin{pmatrix}A_{1,k}\\ \vdots\end{pmatrix}$$

$\boxed{A_{m,k}}$

$\therefore M(T)_{\cdot,k} = M(Tv_k)$

3.65 Linear maps act like matrix multiplication

Suppose $T \in \mathcal{L}(V,W)$ and $v \in V$. Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W.

Then

$$M(Tv) = M(T)M(v)$$

Proof:

Since $v_1, \dots, v_n$ is a basis of V, we can write every $v \in V$ uniquely as

$$v = c_1v_1 + \dots + c_nv_n$$

for some $c_1, \dots, c_n \in \mathbb{F}$. Then we have

$$\begin{aligned} Tv &= T(c_1v_1 + \dots + c_nv_n) \\ &= T(c_1v_1) + \dots T(c_nv_n) \\ &= c_1Tv_1 + \dots + c_nTv_n \end{aligned}$$

Therefore, as $(M(T))_{\cdot,1}, \dots, (M(T))_{\cdot,n}$ is a basis of $\mathbb{F}^{m,n}$

$$\begin{aligned} M(Tv) &= M(c_1Tv_1 + \dots + c_nTv_n) \\ &= M(c_1Tv_1) + \dots + M(c_nTv_n) \\ &= c_1M(Tv_1) + \dots + c_nM(Tv_n) \\ &= c_1(M(T))_{\cdot,1} + \dots + c_n(M(T_v))_{\cdot,n} && \text{by } 3.64 \\ &= M(T)M(V) && \text{by } 3.52 \end{aligned}$$

3.67 Operators

Definition:

· A linear map $T: V \to V$ is called an operator.

· $\mathcal{L}(V)$ denotes the set of all operators on V:

$$\mathcal{L}(V) = \mathcal{L}(V, V)$$

3.69 Injectivity is equivalent to surjectivity in finite dimensions.

Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V)$.

Then the following are equivalent.

a). T is invertible.

b). T is injective.

c). T is surjective.

Proof: (a) implies (b).

Suppose (a) holds, suppose T is invertible. By 3.56 $\Rightarrow$ (b)

(b) implies (c).

Suppose (b) holds, suppose T is injective. By 3.16, $\operatorname{null} T = \{0\}$

By 3.22, we have

$$\begin{aligned} \dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &= \dim V - \dim \{0\} \\ &= \dim V \end{aligned}$$

By 2C.1 we have $\operatorname{range} T = V$. So T is surjective, which is (c).

(c) implies (a).

Suppose (c) holds; suppose T is surjective. Then $\operatorname{range} T = V$

By 3.22. we have

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &= \dim V - \dim V \\ &= 0 = \dim \{0\} \end{aligned}$$

By 2C.1, we have null $T = \{0\}$ By 3.16, T is injective.
So T is both injective and surjective, which is (a).