3.D Invertibility and Isomorphic Vector Spaces

Invertible Linear Maps

3.53 Definition:

- A linear map $T \in \mathcal{L}(V,W)$ is called <u>invertible</u> if there exists a linear map $S \in \mathcal{L}(W,V)$ such that $ST = I_V$ and $TS = I_W$, where I_V and I_W are identity maps on V and W, respective.

Identity Maps

V, W vector spaces
v, w vectors in V, W

$I_V : V \to V$
$I_V(v) = v$

$I_W : W \to W$
$I_W(w) = w$

$ST = I_V$
$T : V \to W$
$S : W \to V$

Therefore
$ST : V \to V$
$(I_V : V \to V)$

If T is invertible with inverse S, then $ST = I_V$ and $TS = I_W$

$TS = I_W$
$S : W \to V$
$T : V \to W$

Therefore $TS : W \to W$
$(I_W : W \to W)$

3.54 Inverse is unique

An invertible linear map has a unique inverse.

Proof: Suppose $T \in \mathcal{L}(V, W)$ is invertible, and let S and $\tilde{S}$ be inverses of T.

Then

$$
\begin{aligned}
S &= S I_W && \text{because } \tilde{S} \text{ is an inverse of } T \\
&= S(T\tilde{S}) \\
&= (ST)\tilde{S} \\
&= I_V \tilde{S} \\
&= \tilde{S} && \text{because } S \text{ is an inverse of } T
\end{aligned}
$$

So $S = \tilde{S}$, which means the inverse of T is unique. ☒

If T is invertible, then it's inverse is denoted by T^{-1}.

If $T \in \mathcal{L}(V, W)$ is invertible, then $T^{-1} \in \mathcal{L}(W, V)$ is the unique element such that

$$T^{-1} T = I_V \text{ and } T T^{-1} = I_W.$$

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3.56 Invertibility is equivalent to surjectivity and injectivity

A linear map $T: V \to W$ invertible if and only if T is injective and surjective.

Proof: Forward direction: If T is invertible, then T is injective & surjective.

Suppose T is invertible. Then its inverse T^{-1} exists. First, we will show that T is injective.
Suppose there exist $u, v \in V$ that satisfy $Tu = Tv$.
Then

$$\begin{aligned} u &= Iu \\ &= (T^{-1}T)u \\ &= T^{-1}(Tu) \\ &= T^{-1}(Tv) \\ &= (T^{-1}T)v \\ &= Iv \\ &= v \end{aligned}$$

So T is injective.

Next, we will show that T is surjective.
Suppose we have an arbitrary vector $w \in W$. (we will argue for all $w \in W$.)
Then we have

$$\begin{aligned} w &= Iw \\ &= (TT^{-1})w \\ &= T(T^{-1}w). \end{aligned}$$

Since $w \in W$ and $T^{-1} \in \mathcal{L}(W, V)$, we have $T^{-1}w \in V$.

So w is of the form Tv for some $v \in V$, and so $w \in$ range T. So $W \subset$ range T. But 3.19 of Axler, range T is a subspace of W. So we have range $T = W$. So T is surjective.

Backward direction: If T is injective and surjective, then T is invertible.

Suppose T is injective and surjective.
For each $w \in W$, (because T is surjective), we can let $Sw \in V$ be a unique (because T is injective) element, that satisfies

$$T(Sw) = w,$$

because T is surjective, $w \in$ range T,

or equivalently

$$(T \circ S)w = w.$$

So $T \circ S = I_W$, where I is the identity map on W.

Next, we need to prove that $S \circ T = I_V$, where I_V is the identity map on V.
We have

$$\begin{aligned} T((S \circ T)v) &= (T \circ S \circ T)v \\ &= (T \circ S)(Tv) \\ &= I_W(Tv) \\ &= Tv \end{aligned}$$

Since T is injective, we get $(S \circ T)v = v$.
In other words, $S \circ T = I_V$.

Finally, we will that $S: W \to V$ is linear (show $S \in \mathcal{L}(W,V)$).

- Additivity: Suppose we have $w_1, w_2 \in W$. Then, since T is linear, we have

$$T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2)$$
$$= w_1 + w_2$$

Since Sw_1 and Sw_2 are unique elements of V that T maps to w_1 and w_2, respectively, it follows that $Sw_1 + Sw_2$ is a unique element of V that T maps to $w_1 + w_2$.

Furthermore, we have

$$S(w_1 + w_2) = S(T(Sw_1 + Sw_2))$$
$$= (S \circ T)(Sw_1 + Sw_2)$$
$$= I_V(Sw_1 + Sw_2)$$
$$= Sw_1 + Sw_2,$$

satisfying additivity

- Homogeneity: Suppose we have $w \in W$ and $\lambda \in \mathbb{F}$. Then since T is linear, we have

$$T(\lambda Sw) = \lambda T(Sw)$$
$$= \lambda w.$$

Since Sw is the unique element of V that T maps to w, it follows that λSw is the unique element of V that T maps to λw.

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Furthermore, we have
$$\begin{aligned} S(\lambda w) &= S(T(\lambda S w)) \\ &= (S \circ T)(\lambda S w) \\ &= I_V(\lambda S w) \\ &= \lambda S w, \end{aligned}$$

satisfying homogeneity.

Therefore, $S \in \mathcal{L}(W, V)$. So T is invertible $\blacksquare$

Isomorphic Vector Space

3.58 Definition:

- An <u>isomorphism</u> is an invertible linear map.
- Two vector spaces V and W are called <u>isomorphic</u> if there exists an isomorphism $T: V \to W$.

3.59 Dimension shows whether vector spaces are isomorphic

Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Then V and W are isomorphic if and only if
$$\dim V = \dim W.$$

Proof:

<u>Forward direction:</u> If V and W are isomorphic, then $\dim V = \dim W$.

Suppose V and W are isomorphic, there exists an isomorphism $T: V \to W$. Since T is isomorphism, it is invertible. By 3.56 of Axler, T is injective and surjective. In other words, we have null $T = \{0\}$ (3.16 of Axler) and range $T = W$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we obtain

$$\begin{aligned} \dim V &= \dim \text{null } T + \dim \text{range } T \\ &= \dim \{0\} + \dim W \\ &= 0 + \dim W \\ &= \dim W. \end{aligned}$$

Backward direction: If $\dim V = \dim W$, then V and W are isomorphic.

Since V and W are finite-dimensional, by 2.32 of Axler, there exist a basis $v_1, .., v_n$ of V and $w_1, .., w_n$ of W, where $n = \dim V = \dim W$.

Define $T: V \to W$ by

arbitrary element of $W \in W$.

$$T(\underbrace{c_1 v_1 + .. + c_n v_n}_{\in V}) = c_1 w_1 + .. + c_n w_n.$$

Then T is linear and well-defined, according to the proof of 3.5 of Axler.

Since $w_1, .., w_n$ is a basis of W, it spans W. Since every vector in W can be written uniquely as $c_1 w_1 + .. + c_n w_n$, T is surjective.

Since $w_1, \ldots, w_n$ is a basis of W, it is linearly independent. In other words, if $c_1, \ldots, c_n \in \mathbb{F}$ satisfy

$$c_1 w_1 + \ldots + c_n w_n = 0,$$

then $$c_1 = 0, \ldots, c_n = 0.$$

Consequently,

$$c_1 v_1 + \ldots + c_n v_n = 0 v_1 + \ldots + 0 v_n$$
$$= 0.$$

Suppose $c_1 v_1 + \ldots + c_n v_n \in \text{null } T$. Then
$$T(c_1 v_1 + \ldots + c_n v_n) = 0,$$
or $$c w_1 + \ldots + c_n w_n = 0.$$

Therefore, $\text{null } T \subseteq \{0\}$. Also, since $T(0) = 0$, we have $\{0\} \in \text{null } T$. Therefore, $\text{null } T = \{0\}$. By 3.16 of Axler, T is injective.

Finally, as T is both injective and surjective. By 3.56, T is an isomorphism.

Note that, if $n = \dim V$, then $\dim V = n = \dim \mathbb{F}^n$

And 3.59 of Axler says that V is isomorphic to $\mathbb{F}^n$

3.60 $\mathcal{L}(V,W)$ and $\mathbb{F}^{m,n}$ are isomorphic

Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W.
Then $\mathcal{M}: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is an isomorphism

Proof: From section 3.C of Axler, $\mathcal{M}: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is linear. We will prove that $\mathcal{M}$ is injective and surjective.

First, we will show that $\mathcal{M}$ is injective.
Suppose $T \in \text{null } \mathcal{M}$. Then $\mathcal{M}(T) = 0$. So we have

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

In other words, $A_{j,k} = 0$ for each $j = 1, \dots, m$ and $k = 1, \dots, n$. Therefore, for each $k = 1, \dots, n$

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j \text{ by Def. 3.32 of Axler}$$

$$= \sum_{j=1}^{n} 0 w_j$$

$$= 0$$

Since $v_1, \dots, v_n$ is a basis of V, we conclude $T = 0$. So $\text{null } \mathcal{M} \subset \{0\}$, or $\text{null } \mathcal{M} = \{0\}$. By 3.16 of Axler, $\mathcal{M}$ is injective.

Next, we will prove that $\mathcal{M}$ is surjective. Suppose $A \in \mathbb{F}^{m,n}$, which means A is an $m \times n$ matrix. Define $T \in \mathcal{L}(V,W)$ by

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j$$

for each $k = 1, \dots, n$. Then

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} = A.$$

So $A \in \operatorname{range} \mathcal{M}$, and so $\mathbb{F}^{m,n} \subset \operatorname{range} \mathcal{M}$. But 3.19 of Axler, range $\mathcal{M}$ is a subspace of $\mathbb{F}^{m,n}$. So range $\mathcal{M} = \mathbb{F}^{m,n}$, and so $\mathcal{M}$ is surjective.

Therefore, $\mathcal{M} : \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is both injective and surjective. By 3.56 of Axler, $\mathcal{M}$ is invertible.

Since $\mathcal{M}$ is both linear and invertible, it is an isomorphism.

3.61 $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$

Suppose V and W are finite-dimensional vector spaces. Then $\mathcal{L}(V,W)$ is finite dimensional and

$$\dim \mathcal{L}(V,W) = (\dim V)(\dim W).$$

Proof: Suppose $m = \dim V$ and $n = \dim W$.

By 3.60 of Axler, $\mathcal{L}(V,W)$ and $\mathbb{F}^{m,n}$ are isomorphic.

By 3.59 of Axler, $\dim \mathcal{L}(V,W) = \dim \mathbb{F}^{m,n}$.
By 3.40 of Axler, $\dim \mathbb{F}^{m,n} = mn$. So we conclude

$$\begin{aligned}\dim \mathcal{L}(V,W) &= \dim \mathbb{F}^{m,n} \\ &= mn \\ &= (\dim V)(\dim W)\end{aligned}$$

as desired.

Linear Maps Thought of as Matrix Multiplication

3.62 Definition:

Suppose $v \in V$ and $v_1, \dots, v_n$ is a basis of V. Then the matrix of a vector v with respect to this basis is the $n \times 1$ matrix

$$M(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, \dots, c_n \in \mathbb{F}$ satisfy

$$v = c_1 v_1 + \dots + c_n v_n.$$

3.63 Example | $2 - 7x + 0x^2 + 5x^3$

- The matrix of $2 - 7x + 5x^3$ with respect to the standard basis $1, x, x^2, x^3$ of $P_3(\mathbb{R})$ is

$$M(2 - 7x + 5x^3) = \begin{pmatrix} 2 \\ -7 \\ 0 \\ 5 \end{pmatrix}$$

• The matrix of $x \in \mathbb{F}^n$ with respect to the standard basis of $\mathbb{F}^n$ is

$$\mathcal{M}(x) = \mathcal{M}((x_1, \dots, x_n)) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

3.64 $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(Tv_k)$

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. Let $k = 1, \dots, n$. Then the k^{th} column of $\mathcal{M}(T)$ equals $\mathcal{M}(Tv_k)$. In other words,

$$(\mathcal{M}(T))_{\cdot,k} = \mathcal{M}(Tv_k).$$

Proof: Let $A = \mathcal{M}(T)$. Then we have

$$\mathcal{M}(T)_{\cdot,k} = A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

"e_1" "e_n"

and $\mathcal{M}(Tv_k) = \mathcal{M}(A_{1,k}w_1 + \dots + A_{m,k}w_m)$ by Def. 3.32 of Axler

$$= \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

Therefore, $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(Tv_k)$. ☒.

3.65 Linear Maps act like matrix multiplication

Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then

$$M(Tv) = M(T)M(v)$$

Proof: Since $v_1, \ldots, v_n$ is a basis of V, we can write every $v \in V$ uniquely as

$$v = c_1 v_1 + \ldots + c_n v_n$$

for some $c_1, \ldots, c_n \in \mathbb{F}$. Then we have

$$\begin{aligned} Tv &= T(c_1 v_1 + \ldots + c_n v_n) \\ &= T(c_1 v_1) + \ldots + T(c_n v_n) \\ &= c_1 T v_1 + \ldots + c_n T v_n. \end{aligned}$$

Therefore, ~~as $(M(T))_{\cdot,1}, \ldots, (M(T))_{\cdot,n}$ is a basis of [illegible]~~, we have

$$\begin{aligned} M(Tv) &= M(c_1 T v_1 + \ldots + c_n T v_n) \\ &= M(c_1 T v_1) + \ldots + M(c_n T v_n) \\ &= c_1 M(T v_1) + \ldots + c_n M(T v_n) \\ &= c_1 (M(T))_{\cdot,1} + \ldots + c_n (M(T))_{\cdot,n} \end{aligned}$$

by 3.64 of Axler

$$= M(T)M(v)$$ by 3.52 of Axler

Operators:

3.67 Definition:

- A linear map $T: V \to V$ is called an operator.
- $\mathcal{L}(V)$ denotes the set of all operators on V:

$$\mathcal{L}(V) = \mathcal{L}(V,V).$$

3.69 Injectivity is equivalent to surjectivity in finite dimensions

Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is invertible;

(b) T is injective;

(c) T is surjective,

Proof: (a) implies (b):

Suppose (a) holds; suppose T is invertible. By 3.56 of Axler, T is injective, which is (b).

(b) implies (c):

Suppose (b) holds; suppose T is injective. By 3.16 of Axler, null $T = \{0\}$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned} \dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &= \dim V - \dim \{0\} \\ &= \dim V - 0 \\ &= \dim V. \end{aligned}$$

By exercise 2.C.1 of Axler, we have range $T = V$. So T is surjective, which is (c).

(c) implies (a):

Suppose (c) holds; suppose T is surjective. Then range $T = V$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &= \dim V - \dim V \\ &= 0 \\ &= \dim \{0\}\end{aligned}$$

By Exercise 2C.1 of Axler, we have null $T = \{0\}$. By 3.16 of Axler, T is injective, so T is both injective and surjective, which is (a).