3D Invertibility and Isomorphic Vector Space

Invertible Linear Maps

Defn 3.53 · A linear map $T \in \mathcal{L}(V, W)$ is called **invertible** if there exists a linear map $S \in \mathcal{L}(W, V)$ such that $ST = I_V$ and $TS = I_W$, where I_V and I_W are identity maps on V and W respectively.

~~identity maps:~~

$I_V : V \to V$ — V, W vector spaces

$I_V(v) = v$ — v, w vectors in V, W.

$I_W : W \to W$

$I_W(w) = w$

$ST = I_V$

$T: V \to W$

$S: W \to V$

Therefore, $ST: V \to V$

$(I_V : V \to V)$

$TS = I_W$

$S : W \to V$

$T : V \to W$

Therefore $TS: W \to W$

$(I_W = W \to W)$

If T is invertible with inverse S, then $ST = I_V$ and $TS = I_W$

Defn 3.54 **Inverse is Unique**

An invertible linear map has a unique inverse.

proof: Suppose $T \in \mathcal{L}(V, W)$ is invertible, and let S and $\tilde{S}$ be inverse of T.

Then
$$\begin{aligned} S &= S I_W \\ &= S(T\tilde{S}) && \text{because } \tilde{S} \text{ is an inverse of } T \\ &= (ST)\tilde{S} \\ &= I_V \tilde{S} && \text{because } S \text{ is an inverse of } T \\ &= \tilde{S} \end{aligned}$$

So $S = \tilde{S}$, which means the inverse of T is unique.

If T is invertible, then its inverse is denoted by T^{-1}

If $T \in \mathcal{L}(V, W)$ is invertible, then $T^{-1} \in \mathcal{L}(W, V)$ is the unique element such that $T^{-1}T = I_V$ and $TT^{-1} = I_W$

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Def 3.56 Invertibility is equivalent to surjectivity and injectivity

A linear map $T: V \to W$ invertible if and only if it is (1) injective and surjective.

Proof. Forward Direction:

If T is invertible, then T is injective & surjective.

Suppose T is invertible. Then its inverse T^{-1} exists.

First, we will show that T is injective.

Suppose there exist $u, v \in V$ that satisfy $Tu = Tv$.

Then
$$\begin{aligned} u &= Iu \\ &= (T^{-1}T)u \\ &= T^{-1}(Tu) \\ &= T^{-1}(Tv) \\ &= (T^{-1}T)v \\ &= Iv \\ &= v \end{aligned}$$
So T is injective.

Next, we will show that T is surjective.

Suppose we have an arbitrary vector $w \in W$. (we will argue for all $w \in W$)

Then we have
$$\begin{aligned} w &= Iw \\ &= (TT^{-1})w \\ &= T(T^{-1}w) \end{aligned}$$
Since $w \in W$ and $T^{-1} \in \mathcal{L}(W, V)$, we have $T^{-1}w \in V$.

So w is of the form Tv for some $v \in V$, and so $w \in \operatorname{range} T$.

So $W \subset \operatorname{range} T$. But 3.19 of Axler, range T is a subspace of W. So we have a range $T = W$.

So T is surjective.

(2) Backward Direction:

If T is injective and surjective, then T is invertible.

Suppose T is injective and surjective. For each $w \in W$ (because T is surjective), we can let $Sw \in V$ be a unique element (because T is injective) that satisfies
$$T(Sw) = w$$
(because T is surjective, $w \in \operatorname{range} T$)

or equivalently
$$(T \circ S)w = w$$

So $T \circ S = I_W$, where I is the identity map on W.

Next, we need to prove that $S \circ T = I_V$, where I_V is the identity map on V.

we have
$$\begin{aligned} T((S \circ V)v) &= (T \circ S \circ T)v \\ &= (T \circ S)(Tv) \\ &= I_W(Tv) \\ &= Tv \end{aligned}$$

Since T is injective, we get
$$(S \circ T)v = v.$$
In other words, $S \circ T = I_V$.

Finally, we will show that $S: W \to V$ is linear (show $S \in \mathcal{L}(W, V)$)

• additivity: Suppose we have $w_1, w_2 \in W$. Then, since T is linear, we have
$$\begin{aligned} T(Sw_1 + Sw_2) &= T(Sw_1) + T(Sw_2) \\ &= w_1 + w_2 \end{aligned}$$
Since Sw_1 and Sw_2 are unique elements of V that T maps to w_1 and w_2, respectively, it follows that $Sw_1 + Sw_2$ is a unique element of V that maps to $w_1 + w_2$. Furthermore, we have
$$\begin{aligned} S(w_1 + w_2) &= S(T(Sw_1 + Sw_2)) \\ &= (S \circ T)(Sw_1 + Sw_2) \\ &= I_V(Sw_1 + Sw_2) \\ &= Sw_1 + Sw_2 \end{aligned}$$
satisfying additivity.

→

• homogeneity: Suppose we have $w \in W$ and $\lambda \in \mathbb{F}$. Then since T is linear, we have
$$T(\lambda S w) = \lambda T(Sw) = \lambda w.$$
Since Sw is the unique element of V that T maps to w, it follows that $\lambda S w$ is the unique element of V that T maps to λw.

Furthermore, we have
$$\begin{aligned} S(\lambda w) &= S(T(\lambda S w)) \\ &= (S \circ T)(\lambda S w) \\ &= I_V(\lambda S w) \\ &= \lambda S w \end{aligned}$$
satisfying homogeneity.

Therefore, $S \in \mathcal{L}(W, V)$. So T is invertible. $\boxed{}$

Isomorphic Vector Space

Defn 3.58
- An isomorphism is an invertible linear map
- Two vector spaces V and W are called isomorphic if there exists an isomorphism $T: V \to W$.

Defn 3.59 Dimension shows whether vector spaces are isomorphic

Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Then V and W are isomorphic if and only if $\dim V = \dim W$.

proof: ① Forward Direction: If V and W are isomorphic, then $\dim V = \dim W$.

Suppose V and W are isomorphic, there exists an isomorphism $T: V \to W$.

Since T is isomorphism, it is invertible. By 3.56 of Axler, T is injective and surjective.

In other words, we have $\operatorname{null} T = \{0\}$ (3.16 of Axler) and $\operatorname{range} T = W$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we obtain
$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim \{0\} + \dim W \\ &= 0 + \dim W \\ &= \dim W \end{aligned}$$

② Backward Direction: If $\dim V = \dim W$, then V and W are isomorphic

Since V and W are finite-dimensional, by 2.32 of Axler, there exist a basis $v_1, \dots, v_n$ of V and $w_1, \dots, w_n$ of W, where $n = \dim V = \dim W$.

Define $T: V \to W$ by
$$T(\underbrace{c_1 v_1 + \cdots + c_n v_n}_{\in V}) = \underbrace{c_1 w_1 + \cdots + c_n w_n}_{\in W}$$
arbitrary element of W

Then T is linear and well-defined, according to the proof of 3.5 of Axler.

Since $w_1, \dots, w_n$ is a basis of W, it spans W. Since every vector in W can be written uniquely as $c_1 w_1 + \cdots + c_n w_n$, T is surjective.

Suppose $c_1 v_1 + \cdots + c_n v_n \in \operatorname{null} T$. Then $T(c_1 v_1 + \cdots + c_n v_n) = 0$, or, $c_1 w_1 + \cdots + c_n w_n = 0$

Since $w_1, \dots, w_n$ is a basis of W, it is linearly independent. In other words, if $c_1, \dots, c_n \in \mathbb{F}$ satisfy
$$c_1 w_1 + \cdots + c_n w_n = 0,$$
then $c_1 = 0, \dots, c_n = 0$.

Consequently, $c_1 v_1 + \cdots + c_n v_n = 0 v_1 + \cdots + 0 v_n = 0$.

Therefore, $\operatorname{null} T \subset \{0\}$. Also, since $T(0) = 0$, we have $\{0\} \subset \operatorname{null} T$. Therefore, $\operatorname{null} T = \{0\}$.

By 3.16 of Axler, T is injective.

Finally, as T is both injective and surjective. By 3.56, T is an isomorphism.

Note that, if $n = \dim V$, then
$$\dim V = n = \dim \mathbb{F}^n$$
and 3.59 of Axler says that V is isomorphic to $\mathbb{F}^n$

Defn 3.60 $\mathcal{L}(V,W)$ and $\mathbb{F}^{m,n}$ are isomorphic

Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. Then $\mathcal{M}: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$.

proof: From section 3.C of Axler, $\mathcal{M}: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is linear. We will prove that $\mathcal{M}$ is injective and surjective.

① First, we will show that $\mathcal{M}$ is injective. Suppose $T \in \operatorname{null} \mathcal{M}$. Then $\mathcal{M}(T) = 0$. So we have

$$\begin{pmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1n} \\ A_{j1} & \cdots & A_{jk} & \cdots & A_{jn} \\ A_{m1} & \cdots & A_{mk} & \cdots & A_{mn} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

In other words, $A_{j,k} = 0$ for each $j = 1, \dots, m$ and $k = 1, \dots, n$. Therefore, for each $k = 1, \dots, n$,

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j \quad \text{by Definition 3.32 of Axler}$$
$$= \sum_{j=1}^{n} 0 w_j$$
$$= 0$$

Since $v_1, \dots, v_n$ is a basis of V we conclude $T = 0$. So $\operatorname{null} \mathcal{M} \subset \{0\}$, or $\operatorname{null} \mathcal{M} = \{0\}$. By 3.16 of Axler, $\mathcal{M}$ is injective.

② Next, we will prove that $\mathcal{M}$ is surjective. Suppose $A \in \mathbb{F}^{m,n}$, which means A is an $m \times n$ matrix. Define $T \in \mathcal{L}(V,W)$ by

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j \quad \text{for each } k = 1, \dots, n. \text{ Then}$$

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{mn} \end{pmatrix} = A.$$

So $A \in \operatorname{range} \mathcal{M}$, and so $\mathbb{F}^{m,n} \subset \operatorname{range} \mathcal{M}$. But 3.19 of Axler, $\operatorname{range} \mathcal{M}$ is a subspace of $\mathbb{F}^{m,n}$. So $\operatorname{range} \mathcal{M} = \mathbb{F}^{m,n}$, and so $\mathcal{M}$ is surjective. Therefore, $\mathcal{M}: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is both injective and surjective. By 3.56 of Axler, $\mathcal{M}$ is invertible. Since $\mathcal{M}$ is both linear and invertible, it is isomorphism.

Defn 3.61 $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$

Suppose V and W are finite-dimensional vector spaces. Then $\mathcal{L}(V,W)$ is finite-dimensional and

$$\dim \mathcal{L}(V,W) = (\dim V)(\dim W).$$

proof: Suppose $m = \dim V$ and $n = \dim W$.

By 3.60 of Axler, $\mathcal{L}(V,W)$ and $\mathbb{F}^{m,n}$ are isomorphic. By 3.59 of Axler, $\dim \mathcal{L}(V,W) = \dim \mathbb{F}^{m,n}$. By 3.40 of Axler, $\dim \mathbb{F}^{m,n} = mn$. So we conclude

$$\dim \mathcal{L}(V,W) = \dim \mathbb{F}^{m,n}$$
$$= mn$$
$$= (\dim V)(\dim W) \quad \text{as desired.}$$

Linear Maps Thought of as Matrix Multiplication

Defn 3.62 Suppose $v \in V$ and $v_1, \dots, v_n$ is a basis of V. Then the matrix of a vector v with respect to this basis is the $n \times 1$ matrix

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where $c_1, \dots, c_n \in \mathbb{F}$ satisfy

$$v = c_1 v_1 + \cdots + c_n v_n$$

Eg 3.63 The matrix of $2 - 7x + 5x^3$ $(2 - 7x + 0x^2 + 5x^3)$ with respect to the standard basis $1, x, x^2, x^3$ of $\mathcal{P}_3(\mathbb{R})$ is

$$\mathcal{M}(2 - 7x + 5x^3) = \begin{pmatrix} 2 \\ -7 \\ 0 \\ 5 \end{pmatrix}$$

• The matrix of $x \in \mathbb{F}^n$ with respect to the standard basis of $\mathbb{F}^n$ is

$$\mathcal{M}(x) = \mathcal{M}((x_1, \dots, x_n)) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$\cancel{(\mathcal{M}(x))}$ ← TYPO in TEXT.

Defn 3.64 $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(Tv_k)$

Suppose $T \in \mathcal{L}(V,W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. Let $k = 1, \dots, n$. Then the k^{th} column of $\mathcal{M}(T)$ equals $\mathcal{M}(Tv_k)$; in other words, $(\mathcal{M}(T))_{\cdot,k} = \mathcal{M}(Tv_k)$

proof: Let $A = \mathcal{M}(T)$. Then we have

$$\mathcal{M}(T)_{\cdot,k} = A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

"c_1" "c_n"

and $\mathcal{M}(Tv_k) = \mathcal{M}(A_{1,k}w_1 + \dots + A_{m,k}w_m)$

by Definition 3.32 of Axler

$$= \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

Therefore, $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(Tv_k)$ $\square$

Defn 3.65 Linear Maps act like matrix multiplication

Suppose $T \in \mathcal{L}(V,W)$ and $v \in V$. Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W.

Then $\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$

Proof: Since $v_1, \dots, v_n$ is a basis of V, we can write every $v \in V$ uniquely as

$$v = c_1v_1 + \dots + c_nv_n$$

for some $c_1, \dots, c_n \in \mathbb{F}$. Then we have

$$\begin{aligned} Tv &= T(c_1v_1 + \dots + c_nv_n) \\ &= T(c_1v_1) + \dots + T(c_nv_n) \\ &= c_1Tv_1 + \dots c_nTv_n \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{M}(Tv) &= \mathcal{M}(c_1Tv_1 + \dots + c_nTv_n) \\ &= \mathcal{M}(c_1Tv_1) + \dots + \mathcal{M}(c_nTv_n) \\ &= c_1\mathcal{M}(Tv_1) + \dots + c_n\mathcal{M}(Tv_n) \\ &= c_1(\mathcal{M}(T))_{\cdot,1} + \dots + c_n(\mathcal{M}(T))_{\cdot,n} \end{aligned}$$

by 3.64 of Axler

$$= \mathcal{M}(T)\mathcal{M}(v).$$ by 3.52 of Axler.

Operators

Defn 3.67 • a linear map $T: V \to V$ is called an operator.

• $\mathcal{L}(V)$ denotes the set of all operators on V.

$$\mathcal{L}(V) = \mathcal{L}(V,V)$$

Thm 3.69 Injectivity is equivalent to surjectivity in finite-dimension

Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is invertable;

(b) T is injective;

(c) T is surjective.

proof: • (a) implies (b):

Suppose (a) holds; suppose T is invertible. By 3.56 of Axler, T is injective.

• (b) implies (c):

Suppose (b) holds; suppose T is injective. By 3.16 of Axler, $\text{null}\,T = \{0\}$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned} \dim \text{range}\,T &= \dim V - \dim \text{null}\,T \\ &= \dim V - \dim\{0\} \\ &= \dim V - 0 \\ &= \dim V. \end{aligned}$$

By Exercise 2.C.1 of Axler, we have range $T = V$. So T is surjective, which is (c). →

• (c) implies (a):

Suppose (c) holds; suppose T is surjective. Then range $T = V$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &= \dim V - \dim V \\ &= 0 \\ &= \dim\{0\}\end{aligned}$$

By Excercise 2.C.1 of Axler, we have $\operatorname{null} T = \{0\}$.
By 3.16 of Axler, T is injective.
So T is both injective and surjective, which is (a).