3D Invertibility & Isomorphic vector spaces

Invertible linear maps

3.53 Def

A linear map $T \in \mathcal{L}(V, W)$ is called invertible if there exists a linear map $S \in \mathcal{L}(W, V)$ such that $ST = I_V$ & $TS = I_W$, where I_V & I_W are identity maps on V & W, respective

identity maps

$I_V: V \to V$ V, W vector spaces

$I_V(v) = v$ v, w vectors in V, W

$I_W: W \to W$ $ST = I_V$ $TS = I_W$

$I_W(w) = w$ $T: V \to W$ $S: W \to V$

$S: W \to V$ $T: V \to W$

Therefore $ST: V \to V$ ($I_V: V \to V$)   Therefore, $TS: W \to W$ ($I_W: W \to W$)

If T is invertible w/ inverse S,
then $ST = I_V$ & $TS = I_W$

3.54 Inverse is unique

An invertible linear map has a unique inverse

Proof: Suppose $T \in \mathcal{L}(V, W)$ is invertible, & let S & $\tilde{S}$ be inverse of T. Then

$$\begin{aligned} S &= SI \\ &= S(T\tilde{S}) && \text{b/c } \tilde{S} \text{ is an inverse of } T \\ &= (ST)\tilde{S} \\ &= I_V \tilde{S} \\ &= \tilde{S} && \text{b/c } S \text{ is an inverse of } T \end{aligned}$$

So $S = \tilde{S}$, which means the inverse of T is unique

~ end of proof ~

If T is invertible, then its inverse is denoted by T^{-1}

If $T \in \mathcal{L}(V, W)$ is invertible, then $T^{-1} \in \mathcal{L}(W, V)$ is the unique element such that

$$T^{-1}T = I_V \text{ \& } TT^{-1} = I_W$$

7/16/19
Tues week 4

3.56 Invertibility is equivalent to surjectivity & injectivity

A linear map $T: V \to W$ is invertible if & only if it T is injective & surjective

Proof: Let $T: V \to W$ be a linear map ($T \in \mathcal{L}(V, W)$)

forward direct.: If T is invertible, then T is injective & surjective

Suppose T is invertible. Then its inverse T^{-1} exists. First, we will show that T is injective. Suppose there exist $u, v \in V$ that satisfy $Tu = Tv$. Then

$$\begin{aligned} u &= Iu \\ &= (T^{-1}T)u \\ &= T^{-1}(Tu) \\ &= T^{-1}(Tv) \\ &= (T^{-1}T)v \\ &= Iv \\ &= v \end{aligned}$$

$\Rightarrow$ so T is injective

Next we will show that T is surjective. Suppose we have an arbitrary vector $w \in W$. (We will argue for all $w \in W$
Then we have

$$\begin{aligned} w &= Iw \\ &= (TT^{-1})w \\ &= T(T^{-1}w) \end{aligned}$$

since $w \in W$ & $T^{-1} \in \mathcal{L}(W, V)$, we have $T^{-1}w \in V$

So w is of the form Tv for some $v \in V$, & so $w \in$ range T.
So $W \subset$ range T. But 3.19 of Axler, range T is a subspace of W.
So we have range $T = W$.

Backward Direct. If T is injective & surjective, then T is invertible.

Suppose T is injective & surjective. For each $w \in W$, we can let $Sw \in V$ be a unique element that satisfies

b/c T is surjective

b/c T is injective

$T(Sw) = w$, because T is surjective, $w \in$ rang

or equivalently

$(T \circ S)$ $w \in W$. So $T \circ S = I_W$, where I_W is the identity map on W.

Next we need to prove that $S \circ T = I_V$, where I_V is the identity map on V. We have

$$\begin{aligned} T((S \circ T)v) &= (T \circ S \circ T)v \\ &= (T \circ S)(Tv) \\ &= I_W(Tv) \\ &= Tv \end{aligned}$$

Since T is injective, we get $(S \circ T)v = v$. In other words, $S \circ T = I_V$

Finally, we will show that $S : W \to V$ is linear (show $S \in \mathcal{L}(W, V)$)

• Add: Suppose we have $w_1, w_2 \in W$. Then, since T is linear, we have

$$\begin{aligned} T(Sw_1 + Sw_2) &= T(Sw_1) + T(Sw_2) \\ &= w_1 + w_2 \end{aligned}$$

since Sw_1 & Sw_2 are unique elements of V that T maps to w_1 & w_2, respectively, it follows that $Sw_1 + Sw_2$ is a unique element of V that T maps to $w_1 + w_2$.

Furthermore, we have

$$S(w_1 + w_2) = S(T(Sw_1 + Sw_2))$$
$$= (S \circ T)(Sw_1 + Sw_2)$$
$$= I_V(Sw_1 + Sw_2)$$
$$= Sw_1 + Sw_2$$
satisfying additivity

- Homogeneity: suppose we have $w \in W$ & $\lambda \in \mathbb{F}$. Then, since T is linear we have $T(\lambda Sw) = \lambda T(Sw)$
$$= \lambda w$$

Since Sw is the unique element of V that T maps to w, it follows that λSw is the unique element of V that T maps to λw. Furthermore, we have
$$S(\lambda w) = S(T(\lambda Sw))$$
$$= (S \circ T)(\lambda Sw)$$
$$= I_V(\lambda Sw)$$
$$= \lambda Sw, \text{ satisfying homogeneity}$$

Therefore, $S \in \mathcal{L}(W, V)$. So T is invertible

Isomorphic Vector Spaces

3.58 Def.

- An isomorphism is an invertible linear map
- Two vector spaces V & W are called isomorphic if there exists an isomorphism $T: V \to W$

3.59 Dimension shows whether vector spaces are isomorphic

Let V & W be finite-dimensional vector spaces over $\mathbb{F}$. Then V & W are isomorphic if & only if $\dim V = \dim W$

Proof: forward direction: If V & W are isomorphic, then $\dim V = \dim W$.

Suppose V & W are isomorphic, there exists an isomorphism $T: V \to W$. Since T is isomorphism, it is invertible. By 3.56 of Axler, T is injective & surjective. In other words, we have null $T = \{0\}$ (3.16 of Axler) & range $T = W$. By the fundamental Thm of linear maps (3.22 of Axler), we obtain $\dim V = \dim \text{null}\, T + \dim \text{range}\, T$
$$= \dim \{0\} + \dim W$$
$$= 0 + \dim W$$
$$= \dim W.$$

Backward direct: If $\dim V = \dim W$, then V & W are isomorphic

Since V & W are finite-dim, by 2.32 of Axler, there exist a basis $v_1, \dots, v_n$ of V & $w_1, \dots, w_n$ of W, where $n = \dim V = \dim W$.

Define $T: V \to W$ by $T(\underbrace{c_1 v_1 + \dots + c_n v_n}_{\in V}) = \underbrace{c_1 w_1 + \dots + c_n w_n}_{\text{arbitrary elem of } w \in W}$.

Then T is linear & well defined, according to the proof of 3.5 of Axler

Since $w_1, \ldots, w_n$ is a basis of W, it spans W. Since every vector in W can be written uniquely as $c_1w_1 + \cdots + c_nw_n$, T is surjective

Since $w_1, \ldots, w_n$ is a basis of W, it is linearly indep. In other words if $c_1, \ldots, c_n \in \mathbb{F}$ satisfy $c_1w_1 + \cdots + c_nw_n = 0$, then $c_1 = 0, \ldots, c_n = 0$.

Suppose $c_1v_1 + \cdots c_nv_n \in \operatorname{null} T$. Then $T(c_1v_1 + \cdots + c_nv_n) = 0$, or $c_1w_1 + \cdots + c_nw_n = 0$

Consequently, $c_1v_1 + \cdots + c_nv_n = 0v_1 + \cdots + 0v_n$
$$= 0$$

Therefore, $\operatorname{null} T \subset \{0\}$. Also, since $T(0) = 0$, we have $\{0\} \subset \operatorname{null} T$. Therefore $\operatorname{null} T = \{0\}$. By 3.16 of Axler, T is injective.

Finally as T is both injective & surjective. By 3.56, T is an isomorphism

Note that if $n = \dim V$, then $\dim V = n = \dim \mathbb{F}^n$ & 3.59 of Axler says that V is isomorphic to $\mathbb{F}^n$

3.60 $\mathcal{L}(V, W)$ & $\mathbb{F}^{m,n}$ are isomorphic

Suppose $v_1, \ldots, v_n$ is a basis of V & $w_1, \ldots, w_n$ is a basis of W. Then $M: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is an isomorphism

Proof: From section 3C of Axler, $M: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is linear. We will prove that M is injective & surjective.

First we will show that M is injective. Suppose $T \in \operatorname{null} M$. Then $M(T) = 0$. So we have
$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

In other words, $A_{j,k} = 0$ for each $j = 1, \ldots, m$ & $k = 1, \ldots, n$. Therefore, for each $k = 1, \ldots, n$,

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j \quad \text{by def 3.32 of Axler}$$

$$= \sum_{j=1}^{m} 0 w_j = 0$$

Since $v_1, \ldots, v_n$ is a basis of V, we conclude $T = 0$. So $\operatorname{null} M \subset \{0\}$, or $\operatorname{null} M = \{0\}$. By 3.16 of Axler, M is injective.

Next we will prove that M is surjective. Suppose $A \in \mathbb{F}^{m,n}$, which means A is an $m \times n$ matrix. Define $T \in \mathcal{L}(V,W)$ by $Tv_k = \sum_{j=1}^{m} A_{j,k} w_j$ for each $k = 1, \ldots, n$. Then

$$M(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = A.$$

So $A \in \operatorname{range} M$, & so $\mathbb{F}^{m,n} \subset \operatorname{range} M$.

But 3.19 of Axler, range M is a subspace of $\mathbb{F}^{m,n}$ so range $M = \mathbb{F}^{m,n}$, & so M is surjective. Therefore, $M: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is both injective & surjective. By 3.56 of Axler, M is invertible. Since M is both linear & invertible, it is an isomorphism

3.61 $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$

Suppose V & W are finite-dim vector spaces. Then $\mathcal{L}(V,W)$ is finite-dim & $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$.

Proof: by 3.60 of Axler, $\mathcal{L}(V,W)$ & $\mathbb{F}^{m,n}$ are isomorphic.

By 3.59 of Axler, $\dim \mathcal{L}(V,W) = \dim \mathbb{F}^{m,n}$

" 3.40 " " , $\dim \mathbb{F}^{m,n}$ so we conclude

$$\dim \mathcal{L}(V,W) = \dim \mathbb{F}^{m,n} = m,n = (\dim V)(\dim W), \text{ as desired}$$

Suppose $m = \dim V$ & $n = \dim W$

linear maps thought of as matrix multiplication

3.62 Def

Suppose $v \in V$ & $v_1, \dots, v_n$ is a basis of V. Then the matrix of a vector v w/respect to this basis is the $n \times 1$ matrix $M(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ where $c_1, \dots, c_n \in \mathbb{F}$ satisfy $v = c_1v_1 + \dots + c_nv_n$

3.63 Example

$2 - 7x + 0x^2 + 5x^3$

- The matrix of $2 - 7x + 5x^3$ to the standard basis $1, x, x^2, x^3$ of $P_3(\mathbb{R})$ is $M(2 - 7x + 5x^3) = \begin{pmatrix} 2 \\ -7 \\ 0 \\ 5 \end{pmatrix}$
- The matrix of $x \in \mathbb{F}^n$ w/respect to the standard basis of $\mathbb{F}^n$ is $M(x) = M((x_1, \dots, x_n)) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

3.64 $M(T)_{\cdot,k} = M(Tv_k)$

Suppose $T \in \mathcal{L}(V,W)$ & $v_1, \dots, v_n$ is a basis of V & $w_1, \dots, w_m$ is a basis of W. Let $k = 1, \dots, n$. Then the k^{th} column of $M(T)$ equals $M(Tv_k)$; in other words, $(M(T))_{\cdot,k} = M(Tv_k)$

Proof: Let $A = M(T)$. Then we have

$$M(T)_{\cdot,k} = A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

& $M(Tv_k) = M(\underbrace{A_{1,k}}_{c_1}w_1 + \dots + \underbrace{A_{m,k}}_{c_n}w_m)$

by def 3.32 of Axler $= \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$

Therefore $M(T)_{\cdot,k} = M(Tv_k)$

3.65 Linear maps act like matrix mult.

Suppose $T \in \mathcal{L}(V, W)$ & $v \in V$. Suppose $v_1, \dots, v_n$ is a basis of V & $w_1, \dots, w_m$ is a basis of W. Then $M(Tv) = M(T)M(v)$.

Proof: Since $v_1, \dots, v_n$ is a basis of V, we can write every $v \in V$ uniquely as $v = c_1v_1 + \dots + c_nv_n$ for some $c_1, \dots, c_n \in \mathbb{F}$. Then we have

$$\begin{aligned} Tv &= T(c_1v_1 + \dots + c_nv_n) \\ &= T(c_1v_1) + \dots + T(c_nv_n) \\ &= c_1Tv_1 + \dots + c_nTv_n \end{aligned}$$

Therefore, we have

$$\begin{aligned} M(Tv) &= M(c_1Tv_1 + \dots + c_nTv_n) \\ &= M(c_1Tv_1) + \dots + M(c_nTv_n) \\ &= c_1M(Tv_1) + \dots + c_nM(Tv_n) \\ &= c_1(M(T))_{\cdot,1} + \dots + c_n(M(Tv_n))_{\cdot,n} \text{ by 3.64 of Axler} \\ &= M(T)M(v) \text{ by 3.52 of Axler} \end{aligned}$$

Operators

3.67 Def

- A linear map $T: V \to V$ is called an operator
- $\mathcal{L}(V)$ denotes the set of all operators on V: $\mathcal{L}(V) = \mathcal{L}(V, V)$

3.69 Injectivity is equivalent to surjectivity in finite-dimensions

Suppose V is a finite-dim vector space & $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is invertible;

(b) T is injective;

(c) T is surjective

Proof: (a) implies (b): Suppose (a) holds; suppose T is invertible. By 3.56 of Axler, T is injective, which is (b).

(b) implies (c): Suppose (b) holds; suppose T is injective. By 3.16 of Axler, null $T = \{0\}$. By the fundamental theorem of linear maps (3.22 of Axler), we have

$$\begin{aligned} \dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &= \dim V - \dim \{0\} \\ &= \dim V - 0 \\ &= \dim V. \end{aligned}$$

By exercise 2C.1 of Axler, we have range $T = V$. So T is surjective, which is (c).

(c) implies (a)

Suppose (c) holds; suppose T is surjective. Then range $T = V$. By the fundamental theorem of linear maps (3.22 of Axler), we have

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &= \dim V - \dim V = 0 = \dim \{0\}. \end{aligned}$$

By exercise 2.C.1 of Axler, we have null $T = \{0\}$. By 3.16 of Axler, T is

3.E Products & Quotients of vector spaces

7/17/19
Wed. week4

3.71 Def

Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbb{F}$

- The product $V_1 \times \cdots \times V_m$ is defined by $V_1 \times \cdots \times V_m = \{(v_1, \ldots, v_m) : v_1 \in V_1, \ldots, v_m \in V_m\}$
- Addition on $V_1 \times \cdots \times V_m$ is defined by $(u_1, \ldots, u_m) + (v_1, \ldots, v_m) = (u_1 + v_1, \ldots, u_m + v_m)$
- Scalar multi. on $V_1 \times \cdots \times V_m$ is defined by $\lambda(v_1, \ldots, v_m) = (\lambda v_1, \ldots, \lambda v_m)$

3.72 Example: $\underbrace{(5 - 6x + 4x^2, (3, 8, 7))}_{\text{length } 2} \in P_2(\mathbb{R}) \times \mathbb{R}^3$

Example: $\underbrace{((1, 2), (3, 4, 5))}_{\text{length } 2} \in \mathbb{R}^2 \times \mathbb{R}^3$

Example: $\underbrace{(1, (2, 3), (4, 5))}_{\text{length } 3} \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$

3.73 Product of vector spaces is a vector space

Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbb{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

Proof: Let $u_i, v_i, w_i \in V_i$ for each $i = 1, \ldots, m$, & let $\lambda \in \mathbb{F}$

- commutativity: Since V_i is a vector space, we have $u_i + v_i = v_i + u_i$ so we have $(u_1, \ldots, u_m) + (v_1, \ldots, v_m) = (u_1 + v_1, \ldots, u_m + v_m) = (v_1 + u_1, \ldots, v_m + u_m) = (v_1, \ldots, v_m) + (u_1, \ldots, u_m)$
- associativity: Since V_i is a vector space, we have $(u_i + v_i) + w_i = u_i (v_i + w_i)$
  so we have
  $((u_1, \ldots, u_m) + (v_1, \ldots, v_m)) + (w_1, \ldots, w_m) = (u_1 + v_1, \ldots, u_m + v_m) + (w_1, \ldots, w_m)$
  $= ((u_1 + v_1) + w_1, \ldots, (u_m + v_m) + w_m)$
  $= (u_1 + (v_1 + w_1), \ldots, u_m + (v_m + w_m))$
  $= (u_1, \ldots, u_m) + (v_1 + w_1, \ldots, v_m + w_m)$
  $= (u_1, \ldots, u_m) + ((v_1, \ldots, v_m) + (w_1, \ldots, w_m))$
- Additive Identity: we have $(0, \ldots, 0) \in V_1 \times \cdots \times V_m$. And it satisfies $(v_1, \ldots, v_m) + (0, \ldots, 0) = (v_1 + 0, \ldots, v_m + 0)$
  $= (v_1, \ldots, v_m)$

  so $(0, \ldots, 0)$ is the additive identity of $V_1 \times \cdots \times V_m$

- **Additive Inverse**: we have $(-v_1, \ldots, -v_m) \in V_1 \times \cdots \times V_m$ & it satisfies
$$(v_1, \ldots, v_m) + (-v_1, \ldots, -v_m) = (v_1 + (-v_1), \ldots, v_m + (-v_m))$$
$$= (v_1 - v_1, \ldots, v_m - v_m)$$
$$= (0, \ldots, 0)$$
so $(-v_1, \ldots, -v_m)$ is the additive inverse of $V_1 \times \cdots \times V_m$.
- **multiplicative Identity**: we have
$$1(v_1, \ldots, v_m) = (1v_1, \ldots, 1v_m)$$
$$= (v_1, \ldots, v_m)$$

- **Distributive prop** for all $a, b \in F$, we have
$$a((u_1, \ldots, u_m) + (v_1, \ldots, v_m)) = a(u_1 + v_1, \ldots, u_m + v_m)$$
$$= (a(u_1 + v_1), \ldots, a(u_m + v_m))$$
$$= (au_1 + av_1, \ldots, au_m + av_m)$$
$$= (au_1, \ldots, au_m) + (av_1, \ldots, av_m)$$
$$= a(u_1, \ldots, u_m) + a(v_1, \ldots, v_m)$$
and
$$(a+b)(v_1, \ldots, v_m) = ((a+b)v_1, \ldots, (a+b)v_m)$$
$$= (av_1 + bv_1, \ldots, av_m + bv_m)$$
$$= (av_1, \ldots, av_m) + (bv_1, \ldots, bv_m)$$
$$= a(v_1, \ldots, v_m) + b(v_1, \ldots, v_m)$$

3.79 Example show that $\mathbb{R}^2 \times \mathbb{R}^3$ is isomorphic to $\mathbb{R}^5$

Note that, as vector spaces, $\mathbb{R}^2 \times \mathbb{R}^3 \neq \mathbb{R}^5$ b/c elements $\underbrace{((x_1, x_2), (x_3, x_4, x_5))}_{\text{length } 2}$ of $\mathbb{R}^2 \times \mathbb{R}^3$ have length 2 but elements $\underbrace{(x_1, x_2, x_3, x_4, x_5)}_{\text{length } 5}$ of $\mathbb{R}^5$ have length 5.

Proof: Define $T: \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$ by $T((x_1, x_2), (x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4, x_5)$

First, we will show that T is injective. Let $((x_1, x_2), (x_3, x_4, x_5)) \in \text{null } T$, which means $T((x_1, x_2), (x_3, x_4, x_5)) = (0, 0, 0, 0, 0)$.

Then we have
$$(0,0,0,0,0) = T((x_1, x_2), (x_3, x_4, x_5))$$
$$= (x_1, x_2, x_3, x_4, x_5)$$
so $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$.

This means $((x_1, x_2), (x_3, x_4, x_5)) = ((0,0), (0,0,0))$

so $\text{null } T \subset \{(0,0), (0,0,0)\}$

$T((0,0), (0,0,0)) = (0,0,0,0,0)$

we also have $\{((0,0), (0,0,0))\} \subset \text{null } T$

Therefore, $\text{null } T = \{((0,0), (0,0,0))\}$

By 3.16 of Axler, T is injective

Next, we will show that T is surjective
For all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, we have
$(x_1, x_2, x_3, x_4, x_5) = T((x_1, x_2), (x_3, x_4, x_5)) \in \text{range } T$
So $\mathbb{R}^5 \subset$ range T. But range T is a subspace of $\mathbb{R}^5$. So we have range $T = \mathbb{R}^5$. So T is surjective
Therefore, by 3.56 of Axler, T is invertible
Next, we will show that T is linear
- Additivity: For all $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}$, we have
$T(((x_1, x_2), (x_3, x_4, x_5)) + ((y_1, y_2), (y_3, y_4, y_5)))$
$= T((x_1+y_1, x_2+y_2), (x_3+y_3, x_4+y_4, x_5+y_5))$
$= (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5)$
$= (x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)$
$= T(((x_1, x_2), (x_3, x_4, x_5))) + T(((y_1, y_2), (y_3, y_4, y_5)))$

- Homogeneity: for all $\lambda \in F$ & for all $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$, we have $T(\lambda((x_1, x_2), (x_3, x_4, x_5))) = T(((\lambda x_1, \lambda x_2), (\lambda x_3, \lambda x_4, \lambda x_5$
$= (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5)$
$= \lambda(x_1, x_2, x_3, x_4, x_5)$
$= \lambda T(((x_1, x_2), (x_3, x_4, x_5)))$

Therefore T is linear
So T is invertible & linear. Therefore, T is an isomorphism

3.75 example
Find a basis of $P_2(\mathbb{R}) \times \mathbb{R}^2$
soln: $((1, (0,0)), (x, (0,0)), (x^2, (0,0)), (0, (1,0)), (0, (0,1))$
$1, x, x^2$ is a basis of $P_2(\mathbb{R})$ $(1,0), (0,1)$ is a basis of $\mathbb{R}^2$

3.76 Dimension of a product is the sum of dimensions
Suppose $V_1, \ldots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dim & $\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V$
Proof: Axler
Choose a basis of each V_j. For each basis vector of each V_j consider the element of $V_1 \times \cdots \times V_m$ that equals the basis vector in the jth slot & 0 in other slots. The list of all such vectors is lin indep. & spans $V_1 \times \cdots \times V_m$. Therefore, it's a basis of $V_1 \times \cdots \times V_m$, w/ length $\dim V_1 + \cdots + \dim V_m$
Ryan's Interpretation:
Let $j = 1, \ldots, m$. Let $v_{j,1}, \ldots, v_{j,n_j}$ be a basis of each V_j. Then $n_j = \dim V_j$ & the ith basis vector of V_j is $v_{j,i}$ for $i = 1, \ldots, n_j$. So we have ?
$(v_{1,1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, v_{1,n_1})$, length: $n_1 = \dim V_1$
$(v_{2,1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, v_{2,n_2})$, length: $n_2 = \dim V_2$
⋮

$(v_{m,1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, v_{m, n_m})$, length: $n_m = \dim V_m$
is a basis of $V_1 \times \cdots \times V_m$

Total length
$n_1 + n_2 + \cdots + n_m$
$= \dim V_1 + \dim V_2 + \cdots + \dim V_m$

Products & Direct Sums

3.77 Products & Direct Sums

Suppose that $U_1, \ldots, U_m$ are subspaces of V.
Define a linear map
$\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$ by $\Gamma(u_1, \ldots, u_m) = u_1 + \cdots + u_m$
Then $U_1 + \cdots + U_m$ is a direct sum if and only if Γ is injective

Proof:

Forward direction If $U_1 + \cdots + U_m$ is a direct sum, then Γ is injective. Suppose $(u_1, \ldots, u_m) \in \text{null}\,\Gamma$, so that
$\Gamma(u_1, \ldots, u_m) = \underbrace{0 + \cdots + 0}_{m}$

Since $U_1 + \cdots + U_m$ is a direct-sum, by 1.44 of Axler, the only way to write the zero vector $0 + \cdots + 0$ is to take $u_1 = 0, \ldots, u_m = 0$. So $(u_1, \ldots, u_m) = 0$ & so $\text{null}\,\Gamma = \{0\}$. By 3.16 of Axler Γ is injective.

Backward direction If Γ is injective, then $U_1 + \cdots + U_m$ is a direct sum. Since Γ is injective, by 3.16 of Axler we have $\text{null}\,\Gamma = \{(\underset{u_1}{0}, \ldots, \underset{u_m}{0})\}$ so the only way to write $0 + \cdots + 0$ is to take $u_1 = 0, \ldots, u_m = 0$. By 1.44 of Axler, $U_1 + \cdots + U_m$ is a direct sum

3.78 A sum is a direct sum if & only if dimensions add up

Suppose V is finite-dim & $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if & only if
$$\dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m$$

Proof: By the proof of 3.77 of Axler, the map $\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$ defined by $\Gamma(u_1, \ldots, u_m) = u_1 + \cdots + u_m$ is surjective. So the fundamental thm of linear maps (3.22 of Axler) gives us

$\dim(U_1 + \cdots + U_m) = \dim \text{range}\,\Gamma$ (b/c Γ is surjective, $\text{range}\,\Gamma = U_1 + \cdots + U_m$)

$= \dim(U_1 \times \cdots \times U_m) - \dim \text{null}\,\Gamma$ by fun. Thm of lin maps

$= \dim(U_1 \times \cdots \times U_m) - \dim\{0\}$ if & only if Γ is injective (3.16 of Axler)

$= \dim(U_1 \times \cdots U_m)$

if & only if Γ is injective.

combine w/ 3.77 & 3.76 of Axler to conclude that $U_1 + \dots + U_m$ is a direct sum if & only if we have

$$\dim(U_1 + \dots + U_m) = \dim(U_1 \times \dots \times U_m)$$

$$= \dim U_1 + \dots + \dim U_m \text{ by 3.76 of Axler}$$