07-15-19

3D Invertibility and Isomorphic Vector Spaces

Invertible Linear Maps

3.53 Definition

A linear map $T \in \mathcal{L}(V,W)$ is called invertible if there exists a linear map $S \in \mathcal{L}(W,V)$ such that $ST = I_V$ and $TS = I_W$, where I_V and I_W are identity maps on V and W respective

Identity maps

$I_V: V \to V$

$I_V(v) = v$

$I_W: W \to W$

$I_W(w) = w$

- V, W vectors spaces
- v, w vectors in V, W

$ST = I_V$

$T: V \to W$

$S: W \to V$

Therefore $ST: V \to V$

$(I_V: V \to V)$

$TS = I_W$

$S: W \to V$

$T: V \to W$

Therefore **TS**$: W \to W$

$(I_W: W \to W)$

If T is invertible with inverse S, then $ST = I_V$ and $TS = I_W$

3.54 Inverse is unique

An invertible linear map has a unique

Proof: Suppose $T \in \mathcal{L}(V,W)$ is invertible, and let S and $\tilde{S}$ be inverse of T. Then

$$
\begin{aligned}
S &= SI_W \\
&= S(T\tilde{S}) \quad \text{because } \tilde{S} \text{ is an inverse of } T \\
&= (ST)\tilde{S} \\
&= I_V\tilde{S} \quad \text{because } S \text{ is an inverse of } T \\
&= \tilde{S}
\end{aligned}
$$

So $S = \tilde{S}$ which means the inverse of T is unique

∎

If T' is invertible, then its inverse is denoted by T^{-1}

If $T \in \mathcal{L}(V, W)$ is invertible, then $T^{-1} \in \mathcal{L}(W, V)$ is the unique element such that

$$T^{-1}T = I_V \quad \text{and} \quad TT^{-1} = I_W$$

$\boxed{\text{07-16-19}}$

3.56 Invertibility is equivalent to surjectivity and injectivity

A linear map $T: V \to W$ invertible if and only if T is injective and surjective

Proof:

Forward direction: If T is invertible, then T is injective & surjective

Suppose T is invertible. Then its inverse T^{-1} exists.

First, we will show that T is injective.

Suppose there exists $u, v \in V$ that satisfy $Tu = Tv$

Then
$$
\begin{aligned}
u &= Iu \\
&= (T^{-1}T)u \\
&= T^{-1}(Tu) = T^{-1}(Tv) \\
&= (T^{-1}T)v \\
&= Iv = v
\end{aligned}
$$

So T is injective

Next, we will show that T is surjective.

Suppose we have an arbitrary vector $w \in W$.

(we will argue for all $w \in W$)

Then we have
$$
\begin{aligned}
w &= Iw \\
&= (TT^{-1})w \\
&= T(T^{-1}w)
\end{aligned}
$$

Since $w \in W$ and $T^{-1} \in \mathcal{L}(W, V)$, we have $T^{-1}w \in V$.

So w is of the form Tv for some $v \in V$, and $w \in \text{range } T$

So $W \subset \text{range } T$. But 3.19 of Axler, range T is a subspace of W

So we have range $T = W$. So T is surjective

Backward direction: If T is injective and surjective, then T is invertible

Suppose T is injective and surjective. For each $w \in W$, we can let

T is surjective

$Sw \in V$ be a unique element that satisfies

T is injective

$$T(Sw) = w$$

T is surjective, $w \in \text{range } T$

or equivalently

$$(T \circ S)w = w$$

So $T \circ S = I_W$, where I is the identity map on W

Next, we need to prove that $S \circ T = I_V$, where I_V is the identity map on V.

we have
$$T((S \circ T)v) = (T \circ S \circ T)v$$
$$= (T \circ S)(Tv)$$
$$= I_W(Tv) = Tv$$

Since T is injective, we get
$$(S \circ T)v = v$$

In other words, $S \circ T = I_V$

Finally, we will show that $S: W \to V$ is linear (show $S \in \mathcal{L}(W, V)$)

Additivity • Suppose we have $w_1, w_2 \in W$. Then, since T is linear, we have
$$T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2)$$
$$= w_1 + w_2$$

Since Sw_1 and Sw_2 are unique elements of V that T maps to w_1 and w_2, respectively it follows that $Sw_1 + Sw_2$ is a unique element of V that T maps to $w_1 + w_2$. Futhermore, we have
$$S(w_1 + w_2) = S(T(Sw_1 + Sw_2))$$
$$= (S \circ T)(Sw_1 + Sw_2)$$
$$= I_V(Sw_1 + Sw_2)$$
$$= Sw_1 + Sw_2$$

satisfying additivity

Homogeneity • Suppose we have $w \in W$ and $\lambda \in \mathbb{F}$. Then since T is linear, we have
$$T(\lambda Sw) = \lambda T(Sw$$
$$= \lambda w$$

Since Sw is the unique element of V that T maps to w, it follows that λSw is the unique element of V that T maps to λw. Futhermore, we have
$$S(\lambda w) = S(T(\lambda Sw))$$
$$= (S \circ T)(\lambda Sw)$$
$$= I_V(\lambda Sw)$$
$$= \lambda Sw,$$

satisfying homogeneity

→ Therefore, $S \in \mathcal{L}(W, V)$. So T is invertible ∎

Isomorphic Vector Spaces

3.58 Definition

- An isomorphism is an invertible linear map.
- Two vector spaces V and W are called isomorphic if there exists an isomorphism $T: V \to W$

3.59 Dimension shows whether vector spaces are isomorphic

Let V and W be finite-dimensional vector spaces over $\mathbb{F}$
Then V and W are isomorphic if and only if $\dim V = \dim W$.

Proof : Forward direction : If V and W are isomorphic, then $\dim V = \dim W$

Suppose V and W are isomorphic, there exists an isomorphism. $T: V \to W$. Since T is isomorphism, it is invertible. By 3.56 of Axler, T is injective and surjective. In other words, we have null $T = \{0\}$ (3.16 of Axler) and range $T = W$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we obtain

$$\begin{aligned}\dim V &= \dim \text{null } T + \dim \text{range } T \\ &= \dim \{0\} + \dim W \\ &= 0 + \dim W = \dim W\end{aligned}$$

Backward direction : If $\dim V = \dim W$, then V and W are isomorphic

Since V and W are finite-dimensional, by 2.32 of Axler, there exist a basis $v_1, \ldots, v_n$ of V and $w_1, \ldots, w_n$ of W, where $n = \dim V = \dim W$

Define $T: V \to W$ by

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$$

Then T is linear and well-defined, according to the proof of 3.5 of Axler. Since $w_1, \ldots, w_n$ is a basis of W, it spans W. Since every vector in W can be written uniquely as $a_1 w_1 + \cdots + a_n w_n$, ~~[illegible]~~

T is surjective.

Suppose $c_1 v_1 + \cdots + c_n v_n \in$ null T. Then $T(c_1 v_1 + \cdots + c_n v_n) = 0$ or $c_1 w_1 + \cdots + c_n w_n = 0$

Since $w_1, \ldots, w_n$ is a basis of W, it is linearly independent.

In other words, if $c_1, \ldots, c_n \in \mathbb{F}$ satisfy

$$c_1 w_1 + \cdots + c_n w_n = 0,$$

then

$$c_1 = 0, \ldots, c_n = 0$$

consequently,

$$c_1 v_1 + \cdots + c_n v_n = 0 v_1 + \cdots 0 v_n = 0$$

Therefore, null $T \subset \{0\}$. Also, since $T(0) = 0$, we have $\{0\} \subset$ null T. Therefore, null $T = \{0\}$. By 3.16 of Axler, T is injective

Finally as T is both injective and surjective. By 3.56, T is an isomorphism

Note that if $n = \dim V$, then

$$\dim V = n = \dim \mathbb{F}^n$$

And 3.59 of Axler says that

$$V \text{ is isomorphic to } \mathbb{F}^n$$

3.60 $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$ are isomorphic

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Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W.

Then $\mathcal{M} : \mathcal{L}(V, W) \to \mathbb{F}^{m,n}$ is an isomorphism

Proof: From Section 3.C of Axler, $\mathcal{M} : \mathcal{L}(V, W) \to \mathbb{F}^{m,n}$ is linear

we will prove that $\mathcal{M}$ is injective and surjective

First, we will show that $\mathcal{M}$ is injective. Suppose $T \in \operatorname{null} \mathcal{M}$.

Then $\mathcal{M}(T) = 0$. So we have

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1k} & \cdots & A_{1n} \\ \vdots & \ddots & & \ddots & \vdots \\ A_{j1} & \cdots & A_{jk} & \cdots & A_{jn} \\ \vdots & & \ddots & & \vdots \\ A_{m1} & \cdots & A_{mk} & \cdots & A_{mn} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

In other words, $A_{j,k} = 0$ for each $j = 1, \dots, m$ and $k = 1, \dots, n$.

Therefore,

$$\begin{aligned} Tv_k &= \sum_{j=1}^{m} A_{j,k} w_j \quad \text{by Definition 3.32 of Axler} \\ &= \sum_{j=1}^{n} 0 w_j \\ &= 0 \end{aligned}$$

Since $v_1, \dots, v_n$ is a basis of V, we conclude $T = 0$. So $\operatorname{null} \mathcal{M} \subset \{0\}$ or $\operatorname{null} \mathcal{M} = \{0\}$. By 3.16 of Axler, $\mathcal{M}$ is injective.

Next we will prove that $\mathcal{M}$ is surjective. Suppose $A \in \mathbb{F}^{m,n}$, which means A is an $m \times n$ matrix. Define $T \in \mathcal{L}(V, W)$ by

$$Tv_k = \sum_{j=1}^{n} A_{j,k} w_j$$

for each $k = 1, \dots, n$. Then

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1n} \\ \vdots & & \\ A_{m1} & \cdots & A_{m,n} \end{pmatrix} = A$$

So $A \in \operatorname{range} \mathcal{M}$, and so $\mathbb{F}^{m,n} \subset \operatorname{range} \mathcal{M}$

But 3.19 of Axler, $\operatorname{range} \mathcal{M}$ is a subspace of $\mathbb{F}^{m,n}$

So $\operatorname{range} \mathcal{M} = \mathbb{F}^{m,n}$, and so $\mathcal{M}$ is surjective

Therefore, $\mathcal{M} : \mathcal{L}(V, W) \to \mathbb{F}^{m,n}$ is both injective and surjective

By 3.56 of Axler, $\mathcal{M}$ is invertible

Since $\mathcal{M}$ is both linear and invertible, it is an isomorphism

3.61 $\dim \mathcal{L}(V,W) = \dim(V)(\dim W)$

Suppose V and W are finite-dimensional vector spaces. Then $\mathcal{L}(V,W)$ is finite-dimensional and

$$\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$$

Proof: Suppose $m = \dim V$ and $n = \dim W$

By 3.60 of Axler, $\mathcal{L}(V,W)$ and $\mathbb{F}^{m,n}$ are isomorphic

By 3.59 of Axler, $\dim \mathcal{L}(V,W) = \dim \mathbb{F}^{m,n}$

By 3.40 of Axler, $\dim \mathbb{F}^{m,n} = mn$. So we conclude

$$\begin{aligned}\dim \mathcal{L}(V,W) &= \dim \mathbb{F}^{m,n} \\ &= mn \\ &= (\dim V)(\dim W)\end{aligned}$$

as desired.

Linear Maps thought of as Matrix Multiplication

3.62 Definition

Suppose $v \in V$ and $v_1, \ldots, v_n$ is a basis of V. Then the ~~matri~~ matrix of a vector v with respect to this basis is the $n \times 1$ matrix

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where $c_1, \ldots, c_n \in \mathbb{F}$ satisfy

$$v = c_1 v_1 + \cdots + c_n v_n$$

3.63 Example

$2 - 7x + 0x^2 + 5x^3$

- The matrix of $2 - 7x + 5x^3$ with respect to the standard basis $1, x, x^2, x^3$ of $\mathcal{P}_3(\mathbb{R})$ is

$$\mathcal{M}(2 - 7x + 5x^3) = \begin{pmatrix} 2 \\ -7 \\ 0 \\ 5 \end{pmatrix}$$

- The matrix of $x \in \mathbb{F}^n$ with respect to the standard basis of $\mathbb{F}^n$ is

$$\mathcal{M}(x) = \mathcal{M}((x_1, \ldots, x_n)) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

3.64 $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(Tv_k)$

Suppose $T \in \mathcal{L}(V,W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Let $k = 1, \ldots, n$. Then the k^{th} column of $\mathcal{M}(T)$ equals $\mathcal{M}(Tv_k)$. In other words,

$$(\mathcal{M}(T))_{\cdot,k} = \mathcal{M}(Tv_k)$$

Proof: Let $A = M(T)$. Then we have

$$M(T)_{\cdot,k} = A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

and

$$M(Tv_k) = M(A_{1,k}w_1 + \cdots + A_{m,k}w_m) \quad \text{by Definition of 3.32 of Axler}$$
$$= \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

therefore, $M(T)_{\cdot,k} = M(Tv_k)$. ∎

3.65 Linear Maps act like matrix multiplication

Suppose $T \in \mathcal{L}(V,W)$ and $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W

Then

$$M(Tv) = M(T)M(v)$$

Proof: Since $v_1, \ldots, v_n$ is a basis of V, we can write every $v \in V$ uniquely as

$$v = c_1v_1 + \cdots + c_nv_n$$

for some $c_1, \ldots, c_n \in \mathbb{F}$. Then we have

$$Tv = T(c_1v_1 + \cdots + c_nv_n)$$
$$= T(c_1v_1) + \cdots + T(c_nv_n)$$
$$= c_1Tv_1 + \cdots + c_nTv_n$$

Therefore, we have

$$M(Tv) = M(c_1Tv_1 + \cdots + c_nTv_n)$$
$$= M(c_1Tv_1) + \cdots + M(c_nTv_n)$$
$$= c_1M(Tv_1) + \cdots + c_nM(Tv_n)$$
$$= c_1(M(T))_{\cdot,1} + \cdots + c_n(M(Tv_n))_{\cdot,n} \quad \text{by 3.64 of Axler}$$
$$= M(T)M(v) \quad \text{by 3.52 of Axler}$$

Operators

3.67 Definition

- A linear map $T: V \to V$ is called an operator
- $\mathcal{L}(V)$ denotes the set of all operators on V.

$$\mathcal{L}(V) = \mathcal{L}(V,V)$$

3.69 Ijectivity is equivalent to surjectivity in finite dimensions

Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V)$

Then the following are equivalent:

(a) T is invertible

(b) T is injective

(c) T is surjective

Proof: (a) implies (b):

Suppose (a) holds; suppose T is invertible. By 3.56 of Axler, T is injective, which is (b)

(b) implies (c):

Suppose (b) holds; suppose T is injective. By 3.16 of Axler null $T = \{0\}$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned} \dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &= \dim V - \dim \{0\} \\ &= \dim V - 0 \\ &= \dim V \end{aligned}$$

by Exercise 2C1 of Axler, we have range $T = V$. So T is surjective, which is (c)

(c) implies (a):

Suppose (c) holds; suppose T is surjective. Then range $T = V$. By the Fundamental Theorem of Linear Maps (3.22 of Axler) we have

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &= \dim V - \dim V \\ &= 0 \\ &= \dim \{0\} \end{aligned}$$

By Exercise 2.C1 of Axler, we have null $T = \{0\}$. By 3.16 of Axler, T is injective,

So T is both injective and surjective, which is (a).