3.D Invertibility and Isomophic Vector Spaces

Invertible Linear Maps

3.53 Definition

- A linear map $T \in \mathcal{L}(V,W)$ is called ~~linear~~ invertible if there exists a linear map $S \in \mathcal{L}(W,V)$ such that $ST = I_V$ and $TS = I_W$, where I_V and I_W are identity maps on V and W, respective.

Identity maps

$I_V : V \to V$.

$I_V(v) = v$

$I_W : W \to W$

$I_W(w) = w$

V, W vector spaces

v, w vectors in V, W

$ST = I_V$

$T: V \to W$

$S: W \to V$

Therefore $ST: V \to V$

$I_V: V \to V$

If T is invertible with inverse S, then $ST = I_V$ and $TS = I_W$.

$TS = I_W$

$S: W \to V$

$T: V \to W$

Therefore, $TS: W \to W$

$(I_W: W \to W)$

3.54 Inverse is unique

An invertible linear map has a unique image.

Proof: Suppose $T \in \mathcal{L}(V,W)$ is invertible. and ~~let~~ let S and $\tilde{S}$ be inverses of T.

Then $S = SI_W$

$= S(T\tilde{S})$ ← because $\tilde{S}$ is an inverse of T.

$= (ST)\tilde{S}$

$= I_V\tilde{S}$ ← because S is an inverse of T.

$= \tilde{S}$

So $S = \hat{S}$, which means the inverse of T is unique.

If T' is invertible, then its inverse is observed by T^{-1}.

If $T \in \mathcal{L}(V, W)$ is invertible, then $T^{-1} \in \mathcal{L}(W, V)$ is the unique element such that $T^{-1}T = I_V$ and $TT^{-1} = I_W$.

3.56 Invertibility is equivalent to surjectivity and injectivity

A linear map $T: V \to W$ invertible if and only if T is injective and surjective.

proof: Forward direction:

If T is invertible, then T is injective & surjective.

Suppose T is invertible. Then its inverse T^{-1} exists.
First, we will show that T is injective.
Suppose there exist $u, v \in V$ that satisfy $Tu = Tv$.
Then

$$\begin{aligned} u &= Iu \\ &= (T^{-1}T)u \\ &= T^{-1}(Tu) \\ &= T^{-1}(Tv) \\ &= (T^{-1}T)v \\ &= Iv \\ &= v. \end{aligned}$$

So T is injective.

Next, we will show that T is surjective.
Suppose we have an arbitrary vector $w \in W$. (we will argue for all $w \in W$)
Then we have

$$\begin{aligned} w &= Iw \\ &= (TT^{-1})w \\ &= T(T^{-1}) \end{aligned}$$

Since $w \in W$ and $T^{-1} \in \mathcal{L}(W, V)$, we have $(T^{-1}w) \in V$.
So w is of the form Tv for some $v \in V$, and so $w \in$ range T.
So $W \subset$ range T. But 3.19 of Axler, range T is a subspace of W.
So we have range $T = W$. So T is surjective.

Backward direction: If T is injective and surjective, then T is invertible.

Suppose T is injective and surjective. For each $w \in W$, we can let
because T is surjective.
$Sw \in V$ be a unique element that satisfies
because T is injective.

$$T(Sw) = w,$$

because T is surjective, $w \in$ range T

or equivalently

$$(T \circ S)_w = w$$

So $T \circ S = I_W$, where I_W is the identity map on W.

Next, we need to prove that $S \circ T = I_V$, where I_V is the identity map on V. We have

$$\begin{aligned} T((S \circ T)v) &= (T \circ S \circ T)v \\ &= (T \circ S)(Tv) \\ &= I_W(Tv) \\ &= Tv \end{aligned}$$

Since T is injective, we get $(S \circ T)v = V$.

In other words, so $T = I_V$.

Finally, we will show that $S: W \to V$ is linear (show $S \in \mathcal{L}(W, V)$)

- Additivity: Suppose we have $w_1, w_2 \in W$. Then, since T is linear, we have

$$\begin{aligned} T(Sw_1 + Sw_2) &= T(Sw_1) + T(Sw_2) \\ &= w_1 + w_2. \end{aligned}$$

Since Sw_1 and Sw_2 are unique elements of V that T maps to w_1 and w_2, respectively, it follows that $Sw_1 + Sw_2$ is a unique element of V that T maps to $w_1 + w_2$.

Furthermore, we have

$$\begin{aligned} S(w_1 + w_2) &= S(T(Sw_1 + Sw_2)) \\ &= (S \circ T)(Sw_1 + Sw_2) \\ &= I_V(Sw_1 + Sw_2) \\ &= Sw_1 + Sw_2. \end{aligned}$$

Satisfying additivity.

• Homogeneity: Suppose we have $w \in W$ and $\lambda \in \mathbb{F}$. Then, since T is linear, we have $T(\lambda S w) = \lambda T(Sw) = \lambda w$.

Since Sw is the unique element of V that T maps to w, it follows that $\lambda S w$ is the unique element of V that T maps to λw. Furthermore, we have
$$\begin{aligned} S(\lambda w) &= S(T(\lambda S w)) \\ &= (S \circ T)(\lambda S w) \\ &= I_V(\lambda S w) \\ &= \lambda S w. \end{aligned}$$
satisfying homogeneity.

Therefore, $S \in \mathcal{L}(W, V)$. So T is invertible.

Isomorphic Vector Spaces

3.58 Definition

- An isomorphism is an invertible linear map.
- Two vector spaces V and W are called isomorphic if there exists an ~~isophort~~ isomorphism $T: V \rightarrow W$.

3.59 Dimension shows whether vector spaces are isomorphic

Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Then V and W are isomorphic if and only if
$$\dim V = \dim W.$$

Proof: Forward direction: If V and W are isomorphic, then $\dim V = \dim W$

Suppose V and W are isomorphic, there exists an isomorphism $T: V \rightarrow W$. Since T is isomorphism, it is invertible. By 3.56 of Axler, T is injective and surjective. In other words, we have null $T = \{0\}$ (3.16 of Axler) and range $T = W$.

Fundamental Theorem of Linear Maps (3.22 of Axler), we obtain

$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim \{0\} + \dim W \\ &= 0 + \dim W \\ &= \dim W \end{aligned}$$

Backward direction: If $\dim V = \dim W$, then V and W are isomorphic.

Since V and W are finite-dimensional, by 2.32 of Axler, there exist a basis $v_1, \dots, v_n$ of V and $w_1, \dots, w_n$ of W, where $n = \dim V = \dim W$.

Define $T: V \to W$ by $T(\underbrace{c_1 v_1 + \cdots + c_n v_n}_{\in V}) = \underbrace{c_1 w_1 + \cdots + c_n w_n}_{\text{arbitrary element of } W} \in W$

Then T is linear and well-defined, according to the proof of 3.5 of Axler. Since $w_1, \dots, w_n$ is a basis of W, it spans W. Since every vector in W can be written uniquely as $c_1 w_1 + \cdots + c_n w_n$, T is surjective.

Suppose $c_1 v_1 + \cdots + c_n v_n \in \operatorname{null} T$, Then

$$T(c_1 v_1 + \cdots + c_n v_n) = 0,$$

or $c_1 w_1 + \cdots + c_n w_n = 0$.

Since $w_1, \dots, w_n$ is a basis of W, it is linearly independent. In other words, if $c_1, \dots, c_n \in \mathbb{F}$ satisfy, $c_1 w_1 + \cdots + c_n w_n = 0$, then $c_1 = 0, \dots, c_n = 0$.

Consequently, $c_1 v_1 + \cdots + c_n v_n = 0 v_1 + \cdots + 0 v_n = 0$.

Therefore, $\operatorname{null} T \subset \{0\}$. Also, since $T(0) = 0$, we have $\{0\} \subset \operatorname{null} T$. Therefore, $\operatorname{null} T = \{0\}$. By 3.16 of Axler, T is injective.

Finally, as T is both injective and surjective. By 3.56, T is an isomophism.

Note that, if $n = \dim V$, then $\dim V = n = \dim \mathbb{F}^n$. And 3.51 of Axler says that V is isomophic to $\mathbb{F}^n$.

3.60 $\mathcal{L}(V,W)$ and $\mathbb{F}^{m,n}$ are isomorphic

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then $\mathcal{M}: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is an isomorphism.

Proof: From Section 3.C of Axler, $\mathcal{M}: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is linear. We will prove that $\mathcal{M}$ is injective and surjective.

First, we will show that $\mathcal{M}$ is injective.

Suppose $T \in \text{null}\, \mathcal{M}$. Then $\mathcal{M}(T) = 0$. So we have

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,k} & \cdots & A_{1,n} \\ \vdots & & \vdots & & \vdots \\ A_{j,1} & \cdots & A_{j,k} & \cdots & A_{j,n} \\ \vdots & & \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,k} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

In other word, $A_{j,k} = 0$ for each $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Therefore, for each $k = 1, \ldots, n$,

$$\begin{aligned} Tv_k &= \sum_{j=1}^{m} A_{j,k} w_j \quad \text{by Definition 3.32 of Axler} \\ &= \sum_{j=1}^{m} 0 w_j \\ &= 0 \end{aligned}$$

Since $v_1, \ldots, v_n$ is a basis of V, we conclude $T = 0$. So $\text{null}\, \mathcal{M} \subset \{0\}$, or $\text{null}\, \mathcal{M} = \{0\}$. By 3.16 of Axler, $\mathcal{M}$ is injective.

Next, we will prove that $\mathcal{M}$ is surjective. Suppose $A \in \mathbb{F}^{m,n}$, which means A is an $m \times n$ matrix. Define $T \in \mathcal{L}(V,W)$ by $Tv_k = \sum_{j=1}^{m} A_{j,k} w_j$ for each $k = 1, \ldots, n$. Then $\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = A$.

So $A \in \text{range}\, \mathcal{M}$, and so $\mathbb{F}^{m,n} \subset \text{range}\, \mathcal{M}$. But 3.19 of Axler, $\text{range}\, \mathcal{M}$ is a subspace of $\mathbb{F}^{m,n}$.

So $\text{range}\, \mathcal{M} = \mathbb{F}^{m,n}$, and so $\mathcal{M}$ is surjective.

Therefore, $\mathcal{M}: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is both injective and surjective.

By 3.56 of Axler, $\mathcal{M}$ is invertible.

Since $\mathcal{M}$ is both linear and invertible, it is an isomorphism.

3.61 $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$

Suppose V and W are finite-dimensional vector spaces. Then $\mathcal{L}(V,W)$ is finite-dimensional and $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$.

Proof. : Suppose $m = \dim V$ and $n = \dim W$.

By 3.60 of Axler, $\mathcal{L}(V,W)$ and $\mathbb{F}^{m,n}$ are isomorphic.

By 3.59 of Axler, $\dim \mathcal{L}(V,W) = \dim \mathbb{F}^{m,n}$.

By 3.40 of Axler, $\dim \mathbb{F}^{m,n} = mn$. So we conclude

$$\begin{aligned}\dim \mathcal{L}(V,W) &= \dim \mathbb{F}^{m,n} \\ &= mn \\ &= (\dim V)(\dim W),\end{aligned}$$ as desired.

Linear Maps Thought of as Matrix Multiplication

3.62 Definition

Suppose $v \in V$ and $v_1, \ldots, v_n$ is a basis of V.

Then the matrix of vector v with respect to this basis is the $n \times 1$ matrix $\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, where $c_1, \ldots, c_n \in \mathbb{F}$ satisfy

$$v = c_1 v_1 + \cdots + c_n v_n.$$

3.63 Example

$2 - 7x + 0x^2 + 5x^3$

- The matrix of $2 - 7x + 5x^3$ with respect to the standard basis $1, x, x^2, x^3$ of $\mathcal{P}_3(\mathbb{R})$ is $\mathcal{M}(2-7x+5x^3) = \begin{pmatrix} 2 \\ -7 \\ 0 \\ 5 \end{pmatrix}$

- The matrix of $x \in \mathbb{F}^n$ with respect to the standard basis of $\mathbb{F}^n$ is

$$\mathcal{M}(x) = \mathcal{M}((x_1, \ldots, x_n)) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

3.64 $\mathcal{M}(T)_{,k} = \mathcal{M}(Tv_k)$

$\cancel{\mathcal{M}(v_k)}$ Typo in the Textbook.

Suppose $T \in \mathcal{L}(V,W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. Let $k = 1, \dots, n$. Then the k^{th} column of $\mathcal{M}(T)$ equals $\mathcal{M}(Tv_k)$; in other words,

$$(\mathcal{M}(T))_{,k} = \mathcal{M}(Tv_k).$$

Proof: Let $A = \mathcal{M}(T)$. Then we have $\mathcal{M}(T)_{\cdot,k} = A_{\cdot,k} = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$

and $\mathcal{M}(Tv_k) = \mathcal{M}(\overset{= c_1 \text{''}}{A_{1,k}} w_1 + \cdots + \overset{= c_n \text{''}}{A_{m,k}} w_m)$ by Definition 3.32 of Axler

$$= \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix}$$

Therefore, $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(Tv_k)$.

3.65 Linear Maps act like matrix multiplication

Suppose $T \in \mathcal{L}(V,W)$ and $v \in V$. Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. Then $\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$

Proof: Since $v_1, \dots, v_n$ is a basis of V, we can write every $v \in V$ uniquely as $v = c_1 v_1 + \cdots + c_n v_n$.
for some $c_1, \dots, c_n \in \mathbb{F}$. Then we have

$$\begin{aligned} Tv &= T(c_1 v_1 + \cdots + c_n v_n) \\ &= T(c_1 v_1) + \cdots + T(c_n v_n) \\ &= c_1 Tv_1 + \cdots + c_n Tv_n. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
M(Tv) &= M(c_1 Tv_1 + \cdots + c_n Tv_n) \\
&= M(c_1 Tv_1) + \cdots + M(c_n Tv_n) \\
&= c_1 M(Tv_1) + \cdots + c_n M(Tv_n) \\
&= c_1(M(T))_{\cdot,1} + \cdots + c_n(M(Tv_n))_{\cdot,n} \quad \text{by 3.60 of Axler} \\
&= M(T)M(v) \quad \text{by 3.52 of Axler.}
\end{aligned}$$

Operators

3.67 Definition

- A linear map $T: V \rightarrow V$ is called an operator.
- $\mathcal{L}(V)$ denotes the set of all operators on V: $\mathcal{L}(V) = \mathcal{L}(V, V)$.

3.69 Injectivity is equivalent to surjectivity in finite-dimensions

Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is invertible.

(b) T is injective

(c) T is surjective

Proof: (a) implies (b)

Suppose (a) holds; suppose T is invertible. By 3.56 of Axler, ~~null T = {0}~~. T is injective, which is (b).

(b) implies (a)

Suppose (b) holds; suppose T is injective. By 3.16 of Axler, null $T = \{0\}$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}
\dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\
&= \dim V - \dim \{0\} \\
&= \dim V - 0 \\
&= \dim V.
\end{aligned}$$

By Exercise 2.C.1 of Axler, we have range $T = V$. So T is surjective, which is (c).

<u>(c) implies (a)</u>:

Suppose (c) holds; suppose T is surjective. Then range $T = V$.
By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &= \dim V - \dim V \\ &= 0 \\ &= \dim \{0\}. \end{aligned}$$

By Exercise 2.C.1 of Axler, we have null $T = \{0\}$. By 3.16 of Axler, T is injective. So T is both injective and surjective, which is (a).