3E. Products and Quotients of Vector Spaces

3.71 product of vector spaces

Suppose $V_1 \dots V_m$ are vector spaces over $\mathbb{F}$

- The product $V_1 \times \cdots \times V_m$ is defined by

  $V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots v_m \in V_m\}$

- Addition on $V_1 \times \cdots \times V_m$ is defined by

  $(u_1, \dots, u_m) + (v_1, \dots v_m) = (u_1 + v_1, \dots, u_m + v_m)$

- Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by

  $\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m)$

3.73 Product of vector spaces is a vector space

Suppose $V_1, \dots V_m$ are vector spaces over $\mathbb{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

Proof: let $u_i, v_i, w_i \in V_i$ for each $i = 1, \dots, m$ and let $\lambda \in \mathbb{F}$

3.74 example Is $\mathbb{R}^2 \times \mathbb{R}^3$ isomorphic to $\mathbb{R}^5$

$T((x_1, x_2), (x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4, x_5)$

First, injective.

let $((x_1, x_2), (x_3, x_4, x_5)) \in \operatorname{null} T$ which means

$T((x_1, x_2), (x_3, x_4, x_5)) = (0, 0, 0, 0, 0)$

We have $(0, 0, 0, 0, 0) = T((x_1, x_2), (x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4, x_5)$

So, $x_1 = \cdots x_5 = 0$

This means $((x_1, x_2), (x_3, x_4, x_5)) = (0, 0, 0, 0, 0)$

So $\operatorname{null} T \subset \{(0, 0, 0, 0, 0)\}$

$T((0, 0), (0, 0, 0)) = (0, 0, 0, 0, 0)$

we also have $\{(0, 0), (0, 0, 0)\} \subset \operatorname{null} T$.

Therefore, $\operatorname{null} T = \{(0, 0), (0, 0, 0)\}$

By 3.16, T is injective

Next, we will show that T is surjective.

For all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, we have $(x_1, x_2, x_3, x_4, x_5) = T((x_1, x_2), (x_3, x_4, x_5)) \in \operatorname{range} T$.

So $\mathbb{R}^5 \subset \operatorname{range} T$. But $\operatorname{range} T$ is a subspace of $\mathbb{R}^5$. So

$\operatorname{range} T = \mathbb{R}^5$, So T is surjective.

Therefore, by 3.56, T is invertible.

Next, we will show that T is linear

- Additivty : For all $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}$

We have $T(((x_1,x_2),(x_3,x_4,x_5)) + ((y_1,y_2),(y_3,y_4,y_5)))$

$= T((x_1+y_1, x_2+y_2),(x_3+y_3, x_4+y_4, x_5+y_5))$

$= (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5)$

$= T((x_1,x_2),(x_3,x_4,x_5)) + T((y_1,y_2),(y_3,y_4,y_5))$

- Homogenerdy: For all $\lambda \in \mathbb{F}$ and for all $x_1,x_2,x_3,x_4,x_5 \in \mathbb{R}$.

$T(\lambda((x_1,x_2),(x_3,x_4,x_5))) = T((\lambda x_1, \lambda x_2),(\lambda x_3, \lambda x_4, \lambda x_5))$

$= (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) = \lambda T((x_1,x_2),(x_3,x_4,x_5))$

Therefore, T is linear.

So, T is linear and invertible

Hence, T is an isomophism.

3.75 Example

Find a basis of $\mathcal{P}_2(\mathbb{R}) \times \mathbb{R}^2$

* Sol$^{\underline{n}}$: $(1,(0,0)), (x,(0,0)), (x^2,(0,0)), (0,(1,0)), (0,(0,1))$

3.76 dim of a product is the sum of dimensions

$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m$

Products and Direct Sums

3.77 Product and Direct Sums

Suppose that $U_1, \dots U_m$ are subspaces of V. Define a linear map

$\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$ by

$\Gamma(u_1, \dots u_m) = u_1 + \cdots + u_m$

Then $U_1 + \cdots + U_m$ is a direct sum iff Γ is injective.

<u>Proof:</u> $(\to)$ if $U_1 + \cdots U_m$ is a direct sum, then Γ is injective.

Suppose $(u_1, \dots u_m) \in \text{null}\,\Gamma$ so that

$\Gamma(u_1, \dots u_m) = (0, 0, \dots 0)$

Since $U_1 + \cdots U_m$ is a direct sum, by 1.44, the only way to write $\vec{0}$ is

to take $u_1 = 0, \dots u_m = 0$.

So $(u_1, \dots u_m) = 0$ and so $\text{null}\,\Gamma = \{0\}$ Hence injective.

($\Leftarrow$) If Γ is injective, then $U_1+\cdots+U_m$ is a direct sum.

Since Γ is injetive, by 3.16 we have null $\Gamma = \{(0,0,\ldots,0)\}$

So the only way to write $0+\cdots+0$ is to take

$u_1=0,\ldots,u_m=0$

By 1.44, $U_1+\cdots+U_m$ is a direct sum.

$\square$

3.78 A sum is a direct sum iff dimensions add up.

Suppose V is finite-dimensional and $U_1,\ldots,U_m$ are subspaces of V.

Then $U_1+\cdots+U_m$ is a direct sum iff

$$\dim(U_1+\cdots+U_m) = \dim U_1+\cdots+\dim U_m$$

Proof: $\Gamma: U_1\times\cdots\times U_m \to U_1+\cdots+U_m$ is defined by

$\Gamma(u_1,\ldots,u_m) = u_1+\cdots+u_m$ is surjective

$\dim \operatorname{range}\Gamma = \dim(U_1+\cdots+U_n)$

So by 3.27, $\dim(U_1+\cdots+U_m) = \dim(U_1\times\cdots\times U_m) - \dim \operatorname{null}\Gamma$

$\dim(U_1+\cdots+U_m) = \dim(U_1\times\cdots\times U_m)$

iff Γ is injective.

Combine 3.77 and 3.76 that $U_1+\cdots U_m$ is a direct sum

iff $\dim(U_1+\cdots+U_m) = \dim(U_1\times\cdots\times U_m)$

$= \dim U_1+\cdots+\dim U_m$

3.78 $\dim(U_1+\cdots+U_m) = \dim U_1+\cdots+\dim U_m$

3.79 $v\in V$ and $U\subset V$ Then $v+U$ is the subset of V

defined by $v+U = \{v+u : u\in U\}$

3.80 ex: Let $V=\mathbb{R}^2$ and $U=\{(x,2x)\in\mathbb{R}^2 : x\in\mathbb{R}\}$

Then U is the line in $\mathbb{R}^2$ thru. the origin with slope 2.

So $(3,1)+U$ is a line in $\mathbb{R}^2$ that contains the point $(3,1)$ and has slope 2, and $(-4,0)+U$ is a line in $\mathbb{R}^2$ that contains the point $(-4,0)$ and has slope 2.

$(3,1)+U = \{(3,1)+(x,2x) : x\in\mathbb{R}\} = \{(3+x, 1+2x) : x\in\mathbb{R}\}$

$(-4,0)+U = \{(-4,0)+(x,2x) : x\in\mathbb{R}\} = \{(-4+x, 2x) : x\in\mathbb{R}\}$

Since $(7,0)$ and $(17,20)$ have the same slope, $(7,0)+U=(17,20)+U$.

Proof: $(17,20)+U=\{(17+x,20+2x).\ x\in\mathbb{R}\}$

$(7,0)+U=\{(7+x,2x):\ x\in\mathbb{R}\}$

$=\{(17-10+x,\ 20-20+2x):x\in\mathbb{R}\}$

$=\{(17+(x-10),\ 20+2(x-10)):x\in\mathbb{R}\}$

Since $x\in\mathbb{R}$, $x-10\in\mathbb{R}$ $\quad=\{(17+y,20+2y):y\in\mathbb{R}\}$

$=(17,0)+U$ Hence, proved.

3.81 • An affine subset of V is a subset of V of the form $v+U$ for some $v\in V$ and some subspace U of V.

• For $v\in V$ and U a subspace of V, the affine subset $v+U$ is said to be parallel to U.

3.82 EX: $U=\{(x,2x):x\in\mathbb{R}^2\}$ $V=\mathbb{R}^2$ as a example 3.80

Then all lines in $\mathbb{R}^2$ with slope of 2 are parallel to U.

• Let $V=\mathbb{R}^3$ and $U=\{(x,y,0)\in\mathbb{R}^3: x,y\in\mathbb{R}\}$

Then the affine subsets of $\mathbb{R}^3$ are all the planes in $\mathbb{R}^3$ that are all the planes in $\mathbb{R}^3$ that are parallel to U. For example, $(0,0,2)+U=\{(x,y,2):\ x,y\in\mathbb{R}\}$ is an affine subset of $\mathbb{R}^3$ and is parallel to U.

3.82 $U\subset V$, quotient space V/U is the set of all affine subsets of V parallel to U.

$$V/U=\{v+U:v\in V\}$$

3.84 Ex: • $U=\{(x,2x)\in\mathbb{R}^2:x\in\mathbb{R}\}$ $\mathbb{R}^2/U$ is the set of all lines in $\mathbb{R}^2$ that have slope 2.

• U is a line in $\mathbb{R}^3$ containing the origin, then $\mathbb{R}^3/U$ is the set of all $\mathbb{R}^3$ parallel to U

i.e. $U_1=\{(x,y,0):x,y\in\mathbb{R}\}$

$\mathbb{R}^3/U_1=\{(0,0,z)+U_1=(x,y,z)\}$

$U_2=\{(0,y,z)\in\mathbb{R}^3:\ y,z\in\mathbb{R}\}$

$\mathbb{R}^3/U_2=\{(x,0,0)+U_2: x,y,z\in\mathbb{R}\}$

3.85 Two affine subsets parallel to U are equal or disjoint.

Suppose U is a subspace of V and $v,w\in V$. Then the following are equivalent.

(a) $v-w\in U$.

(b) $v+U=w+U$.

(c) $(v+U)\cap(w+U)\neq\emptyset$

Proof: Want (a) $\Rightarrow$ (b):

Suppose $v-w \in U$. for $v, w \in V$ and $U \subset V$.

let $u \in U$

we have $(v-w)+u \in U$ closed under addition

then $v+u = w+v-w+u$

$= w+((v-w)+u) \in w+U$.

Therefore $v+U \subset w+U$.

Similarly, $\forall u \in U$,

$w+u = v+w-v+u$

$= v+((-v+w)+u) \in v+U$.

Thus $w+U \subset v+U$

Hence $w+U = v+U$

Want: (b) $\Rightarrow$ (c).

Suppose $v+U = w+U$

Then $\exists\ u \in U$ s.t. $u+v \in v+U$
and $u+v \in w+U$

So $v+u \in (v+U) \cap (w+U)$

Thus $(v+U) \cap (w+U) \neq \emptyset$

Want: (c) $\Rightarrow$ (a)

Suppose $(v+U) \cap (w+U) \neq \emptyset$

then $\exists u_1, u_2 \in U$ s.t. $v+u_1 = w+u_2$

Since $U \subset V$, it is closed under addition and scalar mulp., which means

$u_1 - u_2 \in U$. In fact,

$v-w = u_2 - u_1$

$= -(u_1-u_2) \in U$

3.86 let $U \subset V$.

- Addition: is defined on V/U by $(v+U)+(w+U) = (v+w)+U$
- Scalar multiplication: is defined on V/U by
$\lambda(v+U) = (\lambda v)+U$.

3.87. Quotient space is a vector space

V/U is a vector space.

Proof: let $v, w \in V$ be arbitrary.

first, we need to show that the operations of addition and scalar multiplication make sense on V/U.

Suppose $\hat{v}, \hat{w} \in V$ satisify $v+U = \hat{v}+U$, $w+U = \hat{w}+U$.

First, we will show that addition makes sense on V/U.

Since U is a subspace of V, it is closed under addition, So

$$(v+\hat{v})-(w+\hat{w}) \in U.$$

By 3.85 $$(v+\hat{v})+U = (w+\hat{w})+U$$

Thus addition makes sense on V/U.

Now let $\lambda \in \mathbb{F}$, $\hat{v} \in V$ satisifies $v+U = \hat{v}+U$.

by 3.85 of Axler, $v-\hat{v} \in U$ since $U \subset V$

it is closed under sceler multiplication, which means $\lambda(u-\hat{u}) \in U$

So $$\lambda v - \lambda\hat{v} = \lambda(v-\hat{v}) \in U.$$

by 3.83 $$\lambda v + U = \lambda\hat{v}+U.$$

So, scalar multiplication makes sense on V/U.

Next, we will show that V/U satisifies all exioms of a vector space.

let $v, w, x \in V$, and $\lambda \in \mathbb{F}$.

1. Commutativity: $(v+U)+(w+U) = (v+w)+U = (w+v)+U = (w+U)+(v+U)$
2. Associativity: $((u+U)+(w+U))+(x+U) = ((u+w)+U)+(x+U) = (u+(w+x))+U$
   $= (u+U)+((w+U)+(x+U))$
3. Additivity identity: $(v+U)+(0+U) = (v+0)+U = v+U$
4. Additivity inverse: $(v+U)+((-v)+U) = (v+(-v))+U = 0+U$
5. Multiplicative identity: $1(v+U) = (1v)+U = v+U$.
6. Distributive property: $a((v+U)+(w+U)) = a((v+w)+U) = (a(v+w))+U = (av+aw)+U = (av+U)+(aw+U)$

   and $(a+b)(v+U) = (av+U)+(bv+U) = a(v+U)+b(v+U)$

3.88. quotient map, π.

Suppose $U \subset V$, The quotient map π is the linear map $\pi: V \to V/U$ defined by.

$\pi(v) = v + U$. for $v \in V$.

3.89 $\dim V/U = \dim V - \dim U$.

Proof: let $\pi: V \to V/U$.

Claim: $\text{null } \pi = U$.

Since $v - 0 = v \in U$ so by 3.85 of Axler,

$v + U = 0 + 0$.

In fact, $\pi(v) = v + U = 0 + U$.

so, $v \in \text{null } \pi$, and so $U \subset \text{null } \pi$.

If $v \in \text{null } \pi$, then $\pi(v) = 0 + U$.

Since we also have $\pi(v) = v + 0$,

we conclude, $v + U = 0 + U$.

By 3.85, $v = v - 0 \in U$.

So $\text{null } \pi \subset U$.

Therefore $\text{null } \pi = U$.

Claim 2: $\text{range } \pi = V/U$.

For all, $v \in V/U$.

let $w = \pi(v)$ for some $v \in U$.

In fact, by 3.88

$w = \pi(v)$

$= v + U \in V/U$.

So $\text{range } \pi \subset V/U$.

Suppose $v + U \in V/U$.

By 3.88

$v + U = \pi(v) \in \text{range}(\pi)$

So $V/U \subset \text{range } \pi$.

Thus $\text{range } \pi = V/U$.

By Theorem Fund. $\dim V = \dim \text{null } \pi + \dim \text{range } \pi$

$$\dim V = \dim U + \dim V/U$$
$$\dim V/U = \dim V - \dim U.$$

3.90 $\quad T \in \mathcal{L}(V,W) \quad \tilde{T}: V/(\text{null}\,T) \to W$

$$\tilde{T}(v + \text{null}\,T) = Tv$$

Proof: Suppose $u, v \in V$ satisfy $u + \text{null}\,T = v + \text{null}\,T$.

By 3.85 of Axler, $u - v \in \text{null}\,T$.

So, $T(u-v) = Tu - Tv = 0$.

Thus $Tu = Tv$

Therefore, $\tilde{T}(u + \text{null}\,T) = Tu = Tv = \tilde{T}(v + \text{null}\,T)$

So $\tilde{T}$ is well-defined.

3.91 Null space and range of $\tilde{T}$

Note: $\tilde{T}: V/\text{null}\,T \to W$

Suppose $T \in \mathcal{L}(V,W)$. Then

(a) $\tilde{T}$ is a linear map from $V/(\text{null}\,T)$ to W.

(b) $\tilde{T}$ is injective

(c) $\text{range}\,\tilde{T} = \text{range}\,T$

(d) $V/(\text{null}\,T)$ is isomorphic to $\text{range}\,T$.

Proof: (a)

let $u, v \in V$ and $\lambda \in \mathbb{F}$,

Additivity: $\tilde{T}((u+\text{null}\,T) + (v + \text{null}\,T)) = \tilde{T}((u+v) + \text{null}\,T) = T(u+v) = Tu + Tv = \tilde{T}(u+\text{null}(T)) + \tilde{T}(v+\text{null}\,T)$

Homogeneity: $\tilde{T}(\lambda(v+\text{null}\,T)) = \tilde{T}(\lambda v + \text{null}\,T) = T(\lambda v) = \lambda Tv = \lambda \tilde{T}(v + \text{null}\,T)$

Hence, $\tilde{T} \in \mathcal{L}(V/(\text{null}\,T), W)$

(b) Suppose $v \in V$ satisify $\tilde{T}(v + \text{null}\,T) = 0$

Then $Tv = \tilde{T}(v + \text{null}\,T) = 0$

Thus $v \in \text{null}\,T$.

By 3.85 $\quad v + \text{null}\,T = 0 + \text{null}\,T$.

So $\quad \text{null}\,\tilde{T} \subset \{0 + \text{null}\,T\}$

Since $\tilde{T}$ is linear, $\{0 + \text{null}\,T\} \subset \text{null}\,\tilde{T}$

So $\tilde{T} = \{0 + \text{null}\,T\}$

Thus, $\tilde{T}$ is injective.

(c) suppose $w \in \text{range}\, T$ then $w = Tv$

for some $v \in V$, $w = Tv$

$= \tilde{T}(v + \text{null}\, T) \in \text{range}\, \tilde{T}$

So $\text{range}\, T \subset \text{range}\, \tilde{T}$

Suppose $x \in \text{range}\, \tilde{T}$ Then $x = \tilde{T}(v + \text{null}\, T)$ for some $v \in V$

$x = \tilde{T}(v + \text{null}\, T) = Tv \in \text{range}\, T$

So, $\text{range}\, \tilde{T} \subset \text{range}\, T$.

Therefore $\text{range}\, T = \text{range}\, \tilde{T}$.

(d) By part (c)

$\tilde{T}: V/(\text{null}\, T) \to \text{range}\, T$ is surjective.

So $\tilde{T}$ is surjective.

by part (b), it is invertible.

by part (a), $\tilde{T}$ is an isomorphism

So $V/(\text{null}\, T)$ and $\text{range}\, \tilde{T}$ are isomorphic.

That is $V/(\text{null}\, T)$ and $\text{range}\, T$ are isomorphic.

3E: $\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$

$v + U = \{v + u : u \in U\}$

$V/U = \{v + U : v \in V\}$

$\pi : V \to V/U$.

$\tilde{T} : V/(\text{null}\, T) \to W$ $\quad \tilde{T}(v + \text{null}\, T) = Tv$