3.E Products and Quotients of Vector Spaces

3.71 Def)

Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbb{F}$

- The product $V_1 \times \ldots \times V_m$ is defined by $V_1 \times \ldots \times V_m = \{(v_1, \ldots, v_m) : v_1 \in V_1, \ldots, v_m \in V_m\}$
- Addition on $V_1 \times \ldots \times V_m$ is defined by $(u_1, \ldots, u_m) + (v_1, \ldots, v_m) = (u_1 + v_1, \ldots, u_m + v_m)$
- Scalar multiplication on $V_1 \times \ldots \times V_m$ is defined by $\lambda (v_1, \ldots, v_m) = (\lambda v_1, \ldots, \lambda v_m)$

3.72 Example: $\underbrace{(5 - 6x + 4x^2, (3, 8, 7))}_{\text{length } 2} \in P_2(\mathbb{R}) \times \mathbb{R}^3$

3.73 Product of vector spaces is a vector space

Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbb{F}$. Then $V_1 \times \ldots \times V_m$ is a vector space over $\mathbb{F}$.

Proof: Let $u_i, v_i, w_i \in V_i$ $\exists$ $i = 1, \ldots, m$, & let $\lambda \in \mathbb{F}$

- Commutativity: Since each V_i is a vector space, we have $u_i + v_i = v_i + u_i$.

So we have: $(u_1, \ldots, u_m) + (v_1, \ldots, v_m) = (u_1 + v_1, \ldots, u_m + v_m)$
$= (v_1 + u_1, \ldots, v_m + u_m)$
$= (v_1, \ldots, v_m) + (u_1, \ldots, u_m)$

- Associativity: Since V_i is a vectorspace, we have $(u_i + v_i) + w_i = u_i + (v_i + w_i)$

So we have $((u_1, \ldots, u_m) + (v_1, \ldots, v_m)) + (w_1, \ldots, w_m)$
$= ((u_1 + v_1, \ldots, u_m + v_m) + w_m) = (u_1 + (v_1 + w_1), \ldots, u_m + (v_m + w_m))$
$= (u_1, \ldots, u_m) + (v_1 + w_1, \ldots, v_m + w_m) = (u_1, \ldots, \cancel{v}_m) + ((v_1, \ldots, v_m) + (w_1, \ldots, w_m))$.

- Additive identity: We have $(0, \ldots, 0) \in V_1 \times \ldots \times V_m$. And it satisfies ~~[illegible]~~ $(v_1, \ldots, v_m) + (0, \ldots, 0) = (v_1 + 0, \ldots, v_m + 0)$
$= (v_1, \ldots, v_m)$

So $(0, \ldots, 0)$ is the additive identity of $V_1 \times \ldots \times V_m$.

• Additive Inverse: We have $(-v_1, \dots, -v_m) \in V_1, \dots, V_m$.
And it satisfies $(v_1, \dots, v_m) + (-v_1, \dots, -v_m)$
$= (v_1 + (-v_1), \dots, v_m + (-v_m)) = (v_1 - v_1, \dots, v_m - v_m)$
$= (0, \dots, 0)$
So $(-v_1, \dots, -v_m)$ is the additive inverse of $V_1 \times \dots \times V_m$

Multiplicative Identity: We have $1(v_1, \dots, v_m) =$
$(1v_1, \dots, 1v_m) = v_1, \dots, v_m)$
Distributive properties: $\forall a, b \in \mathbb{F}$, we have
$a((u_1, \dots, u_m) + (v_1, \dots, v_m)) = a(u_1 + v_1, \dots, u_m + v_m)$
$= (a(u_1 + v_1), \dots, a(u_m + v_m))$
$= (au_1 + av_1, \dots, au_m + av_m)$
$= (au_1, \dots, au_m) + (av_1, \dots, av_m)$
$= a(u_1, \dots, u_m) + a(v_1, \dots, v_m)$
and $(a+b)(v_1, \dots, v_m) = ((a+b)v_1, \dots, (a+b)v_m)$
$(av_1 + bv_1, \dots, av_m + bv_m) = (av_1, \dots, av_m) + (bv_1, \dots, bv_m)$
$= a(v_1, \dots, v_m) + b(v_1, \dots, v_m)$

3.72 Example
Show that $\mathbb{R}^2 \times \mathbb{R}^3$ is isomorphic to $\mathbb{R}^5$
Note that, as vector spaces, $\mathbb{R}^2 \times \mathbb{R}^3 \neq \mathbb{R}^5$ because elements $((x_1, x_2), (x_3, x_4, x_5))$ of $\mathbb{R}^2 \times \mathbb{R}^3$ have length 2 but elements $(x_1, x_2, x_3, x_4, x_5)$ of $\mathbb{R}^5$ have length 5.
Proof:
Define $T: \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$ by
$T((x_1, x_2)(x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4, x_5)$
First show T is injective.
let $((x_1, x_2), (x_3, x_4, x_5)) \in \operatorname{null} T$, which means
$T((x_1, x_2), (x_3, x_4, x_5)) = (0, 0, 0, 0, 0)$
Then we have $(0, 0, 0, 0, 0) = T((x_1, x_2)(x_3, x_4, x_5))$
$= (x_1, x_2, x_3, x_4, x_5)$

So $x_1 = 0 = x_2 = x_3 = x_4 = x_5$

This means:

$((x_1, x_2), (x_3, x_4, x_5)) = ((0,0), (0,0,0))$

so null $T \in \{(0,0), (0,0,0)\}$

$T(0,0), (0,0,0) = (0,0,0,0,0)$

we also have $\{(0,0), (0,0,0)\} \in$ null T

Therefore, null $T = \{(0,0), (0,0,0)\}$

By 3.16, T is injective.

Next, show that T is surjective.

$\forall (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, we have

$(x_1, x_2, x_3, x_4, x_5) = T((x_1, x_2), (x_3, x_4, x_5)) \in$ range T

So $\mathbb{R}^5 \in$ range T. But range T is a subspace of $\mathbb{R}^5$.

So we have range $T = \mathbb{R}^5$

So T is surjective.

So, by 3.56, T is invertible.

Next, we will show that T is linear.

- Additivity $\forall x_1, \ldots, x_5, y_1, \ldots, y_5 \in \mathbb{R}$

we have:

$T((x_1, x_2), (x_3, x_4, x_5)) + ((y_1, y_2), (y_3, y_4, y_5))$

$= T((x_1 + y_1, x_2 + y_2), (x_3 + y_3, x_4 + y_4, x_5 + y_5))$

$= (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + y_5)$

$= (x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)$

$= T((x_1, x_2), (x_3, x_4, x_5)) + T((y_1, y_2), (y_3, y_4, y_5))$

- Homogeneity: $\forall \lambda \in \mathbb{F}$ and $\forall x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$

$T(\lambda((x_1, x_2), (x_3, x_4, x_5))) = T((\lambda x_1, \lambda x_2), (\lambda x_3, \lambda x_4, \lambda x_5))$

$= (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) = \lambda(x_1, \ldots, x_5)$

$= \lambda T((x_1, x_2), (x_3, x_4, x_5))$

So, T is linear.

So T is invertible & linear.

Therefore, T is an isomorphism ∎

3.75 Example

Find a basis of $P_2(\mathbb{R}) \times \mathbb{R}^2$

Solution: $(1,(0,0)), (x,(0,0)), (x^2,(0,0)), (0,(1,0)), (0,(0,1))$

$1, x, x^2$ is a basis of $P_2(\mathbb{R})$

3.76 Dim. of product is the sum of dims.

Suppose $V_1, \dots, V_m$ are finite-dim vector spaces.
Then $V_1 \times \dots \times V_m$ is finite dim and $\dim(V_1 \times \dots \times V_m)$
$= \dim V_1 + \dots + \dim V_m$.

Proof:

Choose a basis of each V_i. $\exists$ basis vector of each V_i
consider the element of $V_1 \times \dots \times V_m$ that equals
the basis vector in the ith slot & 0 in the other
slots. The list of all such vectors is linearly
independent and spans $V_1 \times \dots \times V_m$. So it is a basis
of $V_1 \times \dots \times V_m$, with length: $\dim V_1 + \dots + \dim V_m$ ∎

Products & Direct Sums

3.77 Products & direct sums

Suppose that $U_1, \dots, U_m$ are subspaces of V.
Define a linear map.

$\Gamma : U_1 \times \dots \times U_m \to U_1 + \dots + U_m$ by

$\Gamma(u_1, \dots, u_m) = u_1 + \dots + u_m$

The $U_1 + \dots + U_m$ is a direct sum IFF Γ is
injective.

Proof: Forward: If $U_1 + \dots + U_m$ is a direct sum,
the Γ is injective. suppose $(u_1, \dots, u_m) \in \operatorname{null} \Gamma$,
so that $\Gamma(u_1, \dots, u_m) = \underbrace{0 + \dots + 0}_{m}$
since $U_1 + \dots + U_m$ is a direct sum, by 1.44, the only
way to write the zero vector $0 + \dots + 0$ is to
take $u_1 = 0, \dots, u_m = 0$.
So $(u_1, \dots, u_m) = 0$ and so $\operatorname{null} \Gamma = \{0\}$. By 3.16, Γ is injective

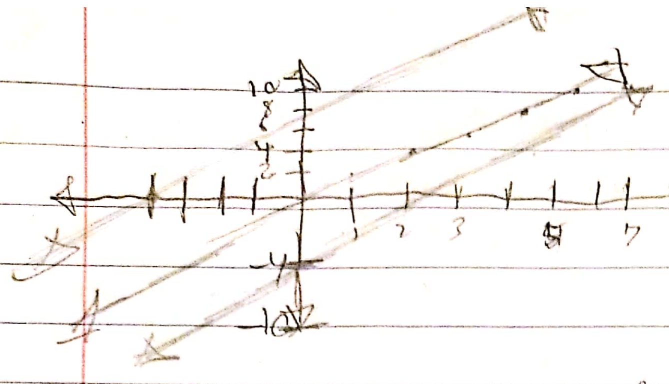
Backward: If Γ is injective, then $U_1 + \cdots + U_m$ is a direct sum. Since Γ is injective, by 3.16, we have null $\Gamma = \{(0, \ldots, 0)\}$
So the only way to write $0 = u_1 + \cdots + u_m$ is to take $u_1 = 0, \ldots, u_m = 0$. By 1.44, $U_1 + \cdots + U_m$ is a direct sum. ∎

3.78 A sum is a direct sum IFF dims add up.

Suppose V is finite-dim and $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum IFF
$$\dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m$$
Proof: By the proof of 3.77, the map $\Gamma: U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$ defined by $\Gamma(u_1, \ldots, u_m) = u_1 + \cdots + u_m$ is surjective.
So the fundamental theorem of linear maps (3.22) gives us $\dim(U_1 + \cdots + U_m) = \dim \text{range}\, \Gamma$
$= \dim(U_1 \times \cdots \times U_m) - \dim \text{null}\, \Gamma$
$= \dim(U_1 \times \cdots \times U_m) - \dim\{0\}$
$= \dim(U_1 \times \cdots \times U_m)$ IFF Γ is injective.
Combine 3.77 & 3.76 to conclude, $U_1 + \cdots + U_m$ is a direct sum IFF, $\dim(U_1 + \cdots + U_m) = \dim(U_1 \times \cdots \times U_m)$
$= \dim U_1 + \cdots + \dim U_m$. by 3.76 ∎

3.E Continued
Quotients of Vector Spaces

3.79 Def
Suppose $v \in V$ and U is a subspace of V. Then $v + U$ is the subset of V defined
$$v + U = \{v + u : u \in U\}$$
3.80 Ex:
Let $V = \mathbb{R}^2$ and $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$
Then U is the line in $\mathbb{R}^2$ through the origin with slope 2.



So $(3,1)+U$ is a line in $\mathbb{R}^2$ that contains the point $(3,1)$ and has slope 2 and $(-4,0)+U$ is a line in $\mathbb{R}^2$ that contains the point $(-4,0)$ and has slope 2

$$(3,1)+U=\{(3,1)+(x,2x):x\in\mathbb{R}\}$$
$$=\{(3+x,1+2x):x\in\mathbb{R}\}$$
$$(-4,0)+U=\{(-4,0)+(x,2x):x\in\mathbb{R}\}$$
$$=\{(-4+x,2x):x\in\mathbb{R}\}$$

Ex: Prove: Since $(8,4)$ and $(17,20)$ lie on the same line, $(7,0)+U=(17,20)+U$.

Proof: $(17,20)+U=\{(17,20)+(x,2x):(x,2x)\in U\}$
$$=\{(17+x,20+2x):x\in\mathbb{R}\}$$
$$(7,0)+U=\{(7,0)+(x,2x):(x,2x)\in U\}$$
$$=\{(7+x,2x):x\in\mathbb{R}\}$$
$$=\{(17-10+x,20-20+2x):x\in\mathbb{R}\}$$
$$=\{(17+(x-10),20+2(x-10)):x\in\mathbb{R}\}$$
$$=\{(17+y,20+2y.\ y\in\mathbb{R}\}$$
$$=\{(17,20)+(y,2y):y\in\mathbb{R}\}$$
$$=(17,20)+U$$

3.81 Def

- An <u>affine subset</u> of V is a subset of V of the form $v+U$ for some $v\in V$ and some subspace U of V.
- If U is a subspace of V, $\forall\, v\in V$, the affine subset $v+U$ is said to be parallel to U.

3.82 Ex:

· let $V = \mathbb{R}^2$ and $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, as in Ex: 3.80. Then all the lines in $\mathbb{R}^2$ with slope 2 are parallel to U. And these lines are affine subsets in $\mathbb{R}^2$.

· let $V = \mathbb{R}^3$ and $U = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$
Then the affine subsets of $\mathbb{R}^3$ are all the planes in $\mathbb{R}^3$ that are parallel to U.

$$(0,0,2) + U = \{(0,0,2) + (x,y,0) : x,y \in \mathbb{R}\}$$
$$= \{(x,y,2) : x,y \in \mathbb{R}\}$$ is an affine subset of $\mathbb{R}^3$ and is parallel to U.

3.83 $\boxed{\text{Def}}$

let U be a subspace of V. Then the quotient space (modulo) V/U is the set of all affine subsets of V parallel to U, written:

$$V/U = \{v + U : v \in V\}$$

example: $(7,0) + U$ is an affine subset of $\mathbb{R}^2$
$$(7,0) + U \in \mathbb{R}^2/U$$

2.84 Ex:

· If $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, then $\mathbb{R}^2/U$ is the set of all lines in $\mathbb{R}^2$ that slope 2.

· If U is a line in $\mathbb{R}^3$ containing the origin, then $\mathbb{R}^3/U$ is the set of all lines in $\mathbb{R}^3$ parallel to U.

For example, $V_1 = \{(x,y,0) \in \mathbb{R}^3 : x,y \in \mathbb{R}\}$.
$$\mathbb{R}^3/V_1 = \{(0,0,z) + V_1 : x,y,z \in \mathbb{R}\}$$
$$V_2 = \{(0,y,z) \in \mathbb{R}^3 : x,y \in \mathbb{R}\},$$
$$\mathbb{R}^3/V_2 = \{x,0,0) + V_2 : x,y,z \in \mathbb{R}\}$$

☆ 3.85 Important ☆ Two affine subsets parallel to U are equal or disjoint

Let U be a subspace of V and $v, w \in V$.
Then the following are equivalent:
(a) $v-w \in U$
(b) $v+U = w+U$
(c) $(v+U) \cap (w+U) \neq \emptyset$.
Proof: (a) implies (b):

(a) implies (b) Suppose (a) holds: $v-w \in U$, let $u \in U$ be arbitrary.
Since U is a subspace of V, in particular it is closed under addition. Since $u \in U$ and $v-w \in U$, we have $(v-w)+u \in U$. $\forall u \in U$, we have $v+u = w+v-w+u \in U$

$$= w+((v-w)+u)$$
$$\in w+U.$$

Also $\forall u \in U$, we have

$$w+u = v+w-v+u \quad \in U$$
$$= v+(-(v-w)+z)$$
$$\in v+U$$

Therefore, $w+U \subset v+U$.
So we conclude the set equality $v+U = w+U$, which is (b).

(b) implies (c) Suppose (b) holds: $v+U = w+U$ then $\exists u \in U$ that then satisfies $v+u \in v+U = w+U$
so $v+u \in (v+U)$ and $v+u \in w+U$.
that is, $v+u \in (v+U) \cap (w+U)$.
So, $(v+U) \cap (w+U) \neq 0$, which is (c).

(c) implies (a) Suppose (c) holds: $(v+U) \cap (w+U) \neq 0$, then $\exists u_1, u_2 \in U$ that satisfies $v+u_1 = w+u_2$
Since U is a subspace of V, it's closed under addition and scalar multiplication, which means $u_1 - u_2 \in U$. In fact, we have $v-w = u_2 - u_1 = -(u_1 - u_2)$
$\in U$,
which is (a) $\square$

3.86 [Def]
Let U be a subspace of V. Then:
· addition: is defined on V/U by $(v+U)+(w+U)$
$= (v+w)+U$
· scalar multiplication: is defined on V/U by
$\lambda(v+U) = (\lambda v)+U$.

3.87 Quotient space is a vector space
Let U be a subspace of V. Then V/U is a vector space respect to the operations defined in [def] 3.86.
Proof: Let $v, w \in V$ be arbitrary.
First we need to show that the operations of addition and scalar multiplication make sense on V/U.
Suppose $\hat{v}, \hat{w} \in V$ satisfy $v+U = \hat{v}+U$ and $w+U = \hat{w}+U$.
First, we will show that addition makes sense on V/U.
Since U is a subspace of V, it's closed under addition.
So $(v+w)-(\hat{v}+\hat{w}) = v-\hat{v}+w-\hat{w} \in U$.
By 3.85 of Axler,
$$(v+w)+U = (\hat{v}+\hat{w})+U.$$
So addition makes sense on V/U.
Now let $\lambda \in \mathbb{F}$ be arbitrary. Suppose $\hat{v} \in V$ satisfies $v+U = \hat{v}+U$.
By 3.85, $v-\hat{v} \in U$. Since U is a subspace of V, it is closed under scalar multiplication, which means $\lambda(v-\hat{v}) \in U$.
So, $\lambda v - \lambda\hat{v} = \lambda(v-\hat{v}) \in U$.
By 3.85, $\lambda v+U = \lambda\hat{v}+U$.
So scalar multiplication makes sense on V/U.
Next, show V/U satisfies all axioms of a vector space.
Let $v, w, x \in V$ and $\lambda, a, b \in \mathbb{F}$ be arbitrary.

· Commutativity: $(v+U)+(w+U)=(v+w)+U$
$$=(w+v)+U = (w+v)+(v+U)$$
· Associativity: $((v+U)+(w+U))+(\lambda+U) = ((v+w)+U)+(\lambda+U)$
$$=((v+w)+\lambda)+U = (v+(w+\lambda))+U = (v+U)+((w+\lambda)+U)$$
$$=(v+U)+((w+U)+(\lambda+U))$$
· Additive identity: $(v+U)+(0+U)=(v+0)+U = v+U$
· Additive inverse: $(v+U)+((-v)+U)=(v+(-v))+U=U$
· Multiplicative identity: $1(v+U)=(1v)+U = v+U$
· Distributive prop: $a((v+U)+(w+U)) = a((v+w)+U)$
$$=a(v+w)+U$$
$$=(av+aw)+u$$
$$=((av+U)+((aw)+U)$$
$$=a(v+U)+a(w+U)$$
and $(a+b)(v+U) = ((a+b)v)+U = (av+bv)+U$
$$=((av)+U)+((bv)+U) = a(v+U)+b(v+U)$$
3.88 Def
let U be a subspace of V. The quotient map is the linear map $\pi: v \to V/U$ defined by
$$\pi(v)=v+u \quad \forall v \in V$$
3.89 Dim. of a quotient space
Suppose V is finite-dim. and U is a subspace of V.
Then, $\dim V/U = \dim V - \dim U$
Proof: let $\pi: V \to V/U$ be the quotient map
First, we claim $\operatorname{null} \pi = U$
Since $v \in U$, we have $v - U = v \in U$, so by 3.85,
$v+U=0+U$.
In fact, we have
$$\pi(v)=v+U=0+U$$
So $v \in \operatorname{null} \pi$, so $U \subset \operatorname{null} \pi$.
If $v \in \operatorname{null} \pi$, then $\pi(v)=0+U$
Since we have $\pi(v)=v+U$,
$v+U=0+U$

By 3.85, $v = v - 0 \in U$.
so null $\pi \subset U$.
Therefore, null $\pi = U$

Next claim: range $\pi = V/U$
will come back to this tomorrow.
By the fundamental theorem of linear maps (3.22), we have
dim V = dim null π + dim range π = dim U + dim V/U
as desired. ∎

range $\pi = V/U$
~~let $w \in$ range π~~
Then $w = \pi(v)$ for some $v \in V$
By def 3.88
We have: $w = \pi(v)$
$= v + U$
$\in V/U$
So we get range $\pi \subset V/U$
So range $\pi = V/U$

Suppose $v + U \in V/U$
By 3.88, $v + U = \pi(v) \in$ range π
so $V/U \subset$ range π

3.90 (Def)
Suppose $T \in \mathcal{L}(V, W)$. Define $\tilde{T}: V/(\text{null } T) \to W$
by $\tilde{T}(v + \text{null } T) = Tv$
Show $\tilde{T}$ makes sense ($\tilde{T}$ is well-defined)
Suppose $u, v \in V$ satisfy
$u + \text{null } T = v + \text{null } T$
By 3.85, $u - v \in \text{null } T$.

This means $T(u-v)=0$

In fact, we have

$$Tu-Tv=T(u-v)=0,$$

so $Tu=Tv$.

So,

$$\tilde{T}(u+\operatorname{null}T)=Tu$$
$$=Tv$$
$$=\tilde{T}(v+\operatorname{null}T),$$

so $\tilde{T}$ is well-defined.

3.91 Null space and range of $\tilde{T}$

Suppose $T\in\mathcal{L}(V,W)$ Then:

a) $\tilde{T}: V/(\operatorname{null}T)\to W$ is a linear map; $\tilde{T}\in\mathcal{L}(V/(\operatorname{null}T),W)$

b) $\tilde{T}$ is injective

c) range $\tilde{T}$ = range T

d) $V/(\operatorname{null}T)$ is isomorphic to range T

Proof: a) let $u,v\in V$ and $\lambda\in\mathbb{F}$

additivity: $\tilde{T}((u+\operatorname{null}T)+(v+\operatorname{null}T))$

$=\tilde{T}((u+v)+\operatorname{null}T)$

$=T(u+v)$

$=Tu+Tv$

$=\tilde{T}(u+\operatorname{null}T)+\tilde{T}(v+\operatorname{null}T)$

homogeneity: $\tilde{T}(\lambda(v+\operatorname{null}T))=\tilde{T}((\lambda v)+\operatorname{null}T)$

$=T(\lambda v)$

$=\lambda Tv=\lambda\tilde{T}(v+\operatorname{null}T)$

So, $\tilde{T}$ is linear.

b) Suppose $v\in V$ satisfies $\tilde{T}(v+\operatorname{null}T)=0$

Then we have

$Tv=\tilde{T}(v+\operatorname{null}T)=0$

So $v-0=v\in\operatorname{null}T$

By 3.85, $v+\operatorname{null}T=0+\operatorname{null}T$

Therefore, null $\tilde{T} \subset \{0 + \text{null } T\}$
But $\tilde{T}(0 + \text{null } T) = 0$ since $\tilde{T}$ is linear.
So $\{0 + \text{null } T\} \subset \text{null } \tilde{T}$
So null $\tilde{T} = \{0 + \text{null } T\}$
By 3.16, $\tilde{T}$ is injective

(c) $\forall v \in V, \tilde{T}(v + \text{null } T) = Tv$
Suppose $w \in \text{range } T$. Then $w = Tv$
for some $v \in V$. In fact,

$$w = Tv$$
$$= \tilde{T}(v + \text{null } T) \in \text{range } \tilde{T}$$

so range $T \subset$ range $\tilde{T}$
Suppose $x \in$ range $\tilde{T}$. Then $x = \tilde{T}(v + \text{null } T)$ for some
$v \in V$. In fact,

$$x = \tilde{T}(v + \text{null } T) = Tv \in \text{range } T$$

so range $\tilde{T} \subset$ range T
So, we conclude range $\tilde{T}$ = range T

(d) By part (c), range $\tilde{T}$ = range T
If we think $\tilde{T}$ as a map into range $\tilde{T}$,
$\tilde{T} : V/(\text{null } T) \to$ range T is surjective
So $\tilde{T}$ is also surjective
By part (b), $\tilde{T}$ is also injective
So by 3.56, $\tilde{T}$ is invertible, by part (a), $\tilde{T}$ is linear.
So, $\tilde{T} : V/(\text{null } T) \to$ range T is an isomorphism.