July. 22

3E

Quotients of Vector Spaces

3.79 Definition

Suppose $v \in V$ and U is a subspace of V.
Then $v+U$ is the subset of V.
defined

$$v+U=\{v+u : u \in U\}$$

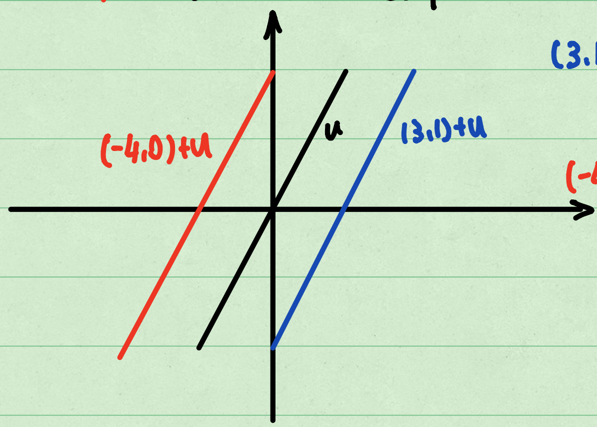
3.80 Example

Let $V=\mathbb{R}^2$ and $U=\{(x,2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$

Then U is the line in $\mathbb{R}^2$ through the origin with slope 2.

So $(3,1)+U$ is a line in $\mathbb{R}^2$ that contains the point $(3,1)$ and has slope 2.

and $(-4,0)+U$ is a line in $\mathbb{R}^2$ that contains the point $(-4,0)$ and has slope 2.



$$(3,1)+U=\{(3,1)+(x,2x) : x \in \mathbb{R}\}$$
$$=\{(3+x,2x+1) : x \in \mathbb{R}\}$$
$$(-4,0)+U=\{(-4,0)+(x,2x) : x \in \mathbb{R}\}$$
$$=\{(-4+x,2x) : x \in \mathbb{R}\}$$

Prove: Since $(7,0)$ and $(17,20)$ lie on the same line,
$(7,0)+U=(17,20)+U$

Proof: $(17,20)+U = \{(17,20)+(x,2x) : (x,2x) \in U\}$

$= \{(17+x,\ 20+2x) : x \in \mathbb{R}\}$

$(7,0)+U = \{(7,0)+(x,2x) : (x,2x) \in U\}$

$= \{(7+x,\ 2x) : x \in \mathbb{R}\}$

$= \{(17-10+x,\ 20-20+2x) : x \in \mathbb{R}\}$

$= \{(17-(10-x),\ 20-2(10-x)) : x \in \mathbb{R}\}$

let $y = 10-x$ $\quad = \{(17+y,\ 20+2y) : y \in \mathbb{R}\}$

Since $x \in \mathbb{R}$ $\quad = \{(17,20)+(y,2y) : y \in \mathbb{R}\}$

$\therefore y \in \mathbb{R}$ $\quad = (17,20)+U$

3.81 Definition

- An affine subset of V is a subset of V of the form $v+U$ for some $v \in V$ and some subspace U of V.
- If U is a subspace of V, for all $v \in V$, the affine subset $v+U$ is said be parallel to U.

3.82 Example

- Let $V = \mathbb{R}^2$ and $U = \{(x, 2x) \in \mathbb{R}^2\}$ as in Example 3.80
Then all the lines in $\mathbb{R}^2$ with slope 2 are parallel to U. And these lines are affine subsets in $\mathbb{R}^2$.
- Let $V = \mathbb{R}^3$ and $U = \{(x, y, 0) \in \mathbb{R}^3, x, y \in \mathbb{R}\}$
Then the affine subsets of $\mathbb{R}^3$ are all the planes in $\mathbb{R}^3$ that are parallel to U.
for example, $(0,0,2)+U = \{(0,0,2)+(x,y,0) : x,y \in \mathbb{R}\}$

$= \{(x,y,2) : x,y \in \mathbb{R}\}$

is an affine subset of $\mathbb{R}^3$ and is parralel to U.

3.83 Definition

Let U be a subspace of V. Then the quotient space V/U, is the set of all affine subsets of V parralel to U.

Written : $V/U = \{v+U : v \in V\}$

3.84 Example:

- If $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, then $\mathbb{R}^2/U$ is the set of all lines in $\mathbb{R}^2$ that have slope 2.
- If U is a line in $\mathbb{R}^3$ containing the origin, then $\mathbb{R}^3/U$ is the set of all lines in $\mathbb{R}^3$, parallel to U.

For example, $U = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$

$\mathbb{R}^3/U_1 = \{(0,0,z) + U : x, y, z \in \mathbb{R}\}$

$U_2 = \{(0, y, z) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$

$\mathbb{R}^3/U_2 = \{(x, 0, 0) + U_2 : x, y, z \in \mathbb{R}\}$

3.85 Two affine subsets parallel to U are equal or disjoint.

Let U be a subspace of V and $v, w \in V$.

Then the following are equivalent:

a). $v - w \in U$

b). $v + U = w + U$

c). $(v+U) \cap (w+U) \neq \emptyset$

Proof: a) implies b)

Suppose a) holds: $v - w \in U$ let $u \in U$. be arbitrary.

Since U is a subspace of V, in particular it is closed under addition. Since $u \in U$ and $v-w \in U$, we have $(v-w)+u \in U$. for all $u \in U$, we have

$$v+u = w+v-w+u \in V$$
$$= w+((v-w)+u) \in w+U$$

$\therefore$ $v+U \subset w+U$

Similarly, for all $u \in U$, we have

$$w+u = v+w-v+u$$
$$= v+\underbrace{(-(v-w)+u)}_{\in U}$$
$$\in v+U$$

$\therefore$ $w+U \subset v+U$

So we conclude the set equality.

$$v+U = w+U. \text{ which is b.}$$

b) implies c)

Suppose b) holds: $v+U = w+U$

Then there exists $u \in U$ that

$$v+u \in v+U = w+U$$

$\therefore$ $v+u \in v+U$ and $v+u \in w+U$.

That is, $v+u \in (v+U) \cap (w+U)$

In other words,

$(v+U) \cap (w+U) \neq \emptyset$, which is c)

c) implies a)

Suppose c) holds: $(v+U) \cap (w+U) \neq \emptyset$. Then there exist $u_1, u_2 \in U$ that satisfies

$$\overset{\in v+U}{v+u_1} = \overset{\in w+U}{w+u_2}$$

Since U is a subspace of V, it is closed under addition and scalar multiplication, which means $u_1-u_2 \in U$. In fact, we have, $v-w = u_2-u_1$

$$= -(u_1-u_2) \in U,$$

which is a).

3.86 Definition

Let U be a subspace of V. Then:

- addition is defined on V/U by

$$(v+U)+(w+U)=(v+w)+U$$

- scalar multiplication is defined on V/U by

$$\lambda(v+U) = (\lambda v)+U$$

3.87 Quotient space is a vector space

Let U be a subspace of V. Then V/U is a vector space with respect to the operations defined in 3.86.

Proof:

Let $v, w \in V$ be arbitrary.

First, we need to show that the operations of addition and scalar multiplication make sense of V/U.

Suppose $\hat{v}, \hat{w} \in V$ satisfy $v+U=\hat{v}+U$ and $w+U=\hat{w}+U$.

First, we will show that addition makes sense on V/U.

since U is a subspace of V, it is closed under addition.

So $(v+\hat{v})+(w-\hat{w}) \in U$.

or $(v-w)-(\hat{v}-\hat{w}) \in U$

By 3.85 $\quad (v+w)+U = (\hat{v}+\hat{w})+U$

So addition makes sense on V/U

Now, let $\lambda \in \mathbb{F}$ be arbitrary. Suppose $\hat{v} \in V$ satisfy $v+U = \hat{v}+U$. By 3.85, $v-\hat{v} \in U$. Since U is a subspace of V, it is closed under scalar multiplication. which means $\lambda(u-\hat{u}) \in V$.

So we have.

$$\lambda v - \lambda \hat{v} = \lambda(v-\hat{v}) \in U$$

By 3.85 $\quad \lambda v + U = \lambda \hat{v} + U$

So scalar multiplication makes sense on V/U.

Next, we will show that V/U satisfies all axioms of a vector space.

Let $v, w, x \in V$ and $\lambda \in \mathbb{F}$.

- Commutativity:
$$\begin{aligned}(v+U)+(w+U) &= (v+w)+U \\ &= (w+v)+U \\ &= (w+U)+(v+U)\end{aligned}$$

- Associativity:
$$\begin{aligned}((v+U)+(w+U))+(x+U) &= ((v+w)+U)+(x+U) \\ &= ((v+w)+x)+U \\ &= (v+(w+x))+U \\ &= (v+U)+((w+x)+U) \\ &= (v+U)+((w+U)+(x+U))\end{aligned}$$

- Additive identity:
$$\begin{aligned}(v+U)+(0+U) &= (v+0)+U \\ &= v+U\end{aligned}$$

- Additive inverse:
$$\begin{aligned}(v+U)+((-v)+U) &= (v+(-v))+U \\ &= 0+U.\end{aligned}$$

- Multiplicative identity: $1(v+U) = (1v)+U$
$$= v+U$$
- Distributive properties: $a((v+U)+(w+U)) = a((v+w)+U)$
$$= a(v+w)+U$$
$$= (av+aw)+U$$
$$= (av+U)+((aw)+U)$$
$$= a(v+U)+a(w+U)$$

and: $(a+b)(v+U) = (a+b)v+U$
$$= (av+bv)+U$$
$$= (av+U)+(bv+U)$$
$$= a(v+U)+b(v+U)$$

3.88 **Definition**

Let U be a subspace of V. The quotient map is the linear map. $\pi: V \to V/U$ defined by

$\pi(v) = v+U$ for all $v \in V$.

3.89 **Dimension of u quotient space**

Suppose V is finite-dimensional and U is a subspace of V. Then $\dim V/U = \dim V - \dim U$

Proof: Let $\pi: V \to V/U$ be the quotient map.

First, we claim $\operatorname{null} \pi = U$.

Since, $v \in U$, we have $v - 0 = v \in U$, so by 3.85.

$v+U = 0+U$

In fact, we have

$$\pi(v) = v + U = 0 + U.$$

So $v \in \operatorname{null} \pi$, and so $U \subset \operatorname{null} \pi$.

If $v \in \operatorname{null} \pi$, then $\pi(v) = 0 + U$.

Since we also have $\pi(v) = v + U$.

We conclude,

$$v + U = 0 + U$$

By 3.85 $v = v - 0 \in U$

So $\operatorname{null} \pi \subset U$.

Therefore, we conclude the set equality $\operatorname{null} \pi = U$.

Next claim, $\operatorname{range} \pi = V/U$

July.23 Let $w \in \operatorname{range} \pi$

Then $w = \pi(v)$ for some $v \in U$

By 3.88.

We have $w = \pi(v)$

$= v + U \in V/U$

$\therefore$ We get $\operatorname{range} \pi \subset V/U$

Suppose we have $v + U \in V/U$

By 3.88

$v + U = \pi(v) \in \operatorname{range} \pi$

So $V/U \subset \operatorname{range} \pi$

$\therefore \operatorname{range} \pi = V/U$