7/17 3.E Products and Quotients of Vector Spaces

3.71 Definition:

Suppose $V_1, .., V_m$ are vector spaces over $\mathbb{F}$

- The <u>product</u> $V_1 \times .. \times V_m$ is defined by
$$V_1 \times .. \times V_m = \{(v_1, .., v_m) : v_1 \in V_1, .., v_m \in V_m\}.$$
- Addition on $V_1 \times .. \times V_m$ is defined by
$$(u_1, .., u_m) + (v_1, .., v_m) = (u_1 + v_1, .., u_m + v_m)$$
- Scalar multiplication on $V_1 \times .. \times V_m$ is defined by
$$\lambda(v_1, .., v_m) = (\lambda v_1, .., \lambda v_m)$$

3.72 Example | $\underbrace{(5-6x+4x^2, (3,8,7))}_{\text{length } 2} \in \mathcal{P}_2(\mathbb{R}) \times \mathbb{R}^3$

Example: $\underbrace{((1,2),(3,4,5))}_{\text{length } 2} \in \mathbb{R}^2 \times \mathbb{R}^3$

Example: $\underbrace{(1,(2,3),(4,5))}_{\text{length } 3} \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$

3.73 Product of vector spaces is a vector space

Suppose $V_1, .., V_m$ are vector spaces over $\mathbb{F}$. Then $V_1 \times .. \times V_m$ is a vector space over $\mathbb{F}$.

Proof: let $u_i, v_i, w_i \in V_i$ for each $i = 1, .., m$, and let $\lambda \in \mathbb{F}$,

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Commutativity: Since each V_i is a vector space, we have $u_i + v_i = v_i + u_i$.

So we have $(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m)$
$= (v_1 + u_1, \dots, v_m + u_m)$
$= (v_1, \dots, v_m) + (u_1, \dots, u_m)$

Associativity: Since [redacted] V_i is a vector space, we have $(u_i + v_i) + w_i = u_i + (v_i + w_i)$.

So we have

$((u_1, \dots, u_m) + (v_1, \dots, v_m)) + (w_1, \dots, w_m)$
$= (u_1 + v_1, \dots, u_m + v_m) + (w_1, \dots, w_m)$
$= ((u_1 + v_1) + w_1 + \dots + (u_m + v_m) + w_m)$
$= (u_1 + (v_1 + w_1) + \dots + u_m + (v_m + w_m))$
$= (u_1, \dots, u_m) + (v_1 + w_1, \dots, v_m + w_m)$
$= (u_1, \dots, u_m) + ((v_1, \dots, v_m) + (w_1, \dots, w_m))$

Additive Identity: we have $(\underbrace{0, \dots, 0}_{\text{zero vectors}}) \in V_1 \times \dots \times V_m$. And we have

$(v_1, \dots, v_m) + (0, \dots, 0) = (v_1 + 0, \dots, v_m + 0)$
$= (v_1, \dots, v_m)$

So $(0, \dots, 0)$ is the additive identity of $V_1 \times \dots \times V_m$.

Additive Inverse: We have $(-v_1, \dots, -v_n) \in V_1 \times \dots \times V_m$. And it satisfies $(v_1, \dots, v_m) + (-v_1, \dots, -v_m) = (v_1 + (-v_1), \dots, v_m + (-v_m))$
$= (v_1 - v_1, \dots, v_m - v_m)$
$= (0, \dots, 0)$

So $(-v_1, \dots, -v_m)$ is the additive inverse of $V_1 \times \dots \times V_m$.

- Multiplicative identity: we have

$$1(v_1,\dots,v_m) = (1v_1,\dots,1v_m) = (v_1,\dots,v_m)$$

- Distributive properties:

For all $a,b \in \mathbb{F}$, we have

$$\begin{aligned} a((u_1,\dots,u_m)+(v_1,\dots,v_m)) &= a(u_1+v_1,\dots,u_m+v_m) \\ &= (a(u_1+v_1),\dots,a(u_m+v_m)) \\ &= (au_1+av_1,\dots,au_m+av_m) \\ &= (au_1,\dots,au_m)+(av_1,\dots,av_m) \\ &= a(u_1,\dots,u_m)+a(v_1,\dots,v_m) \end{aligned}$$

and

$$\begin{aligned} (a+b)(v_1,\dots,v_m) &= ((a+b)v_1,\dots,(a+b)v_m) \\ &= (av_1+bv_1,\dots,av_m+bv_m) \\ &= (av_1,\dots,av_m)+(bv_1,\dots,bv_m) \\ &= a(v_1,\dots,v_m)+b(v_1,\dots,v_m) \end{aligned}$$

3.74 Example | Show that $\mathbb{R}^2 \times \mathbb{R}^3$ is isomorphic to $\mathbb{R}^5$.

Note that, as vector spaces, $\mathbb{R}^2 \times \mathbb{R}^3 \neq \mathbb{R}^5$ because elements $\underbrace{((x_1,x_2),(x_3,x_4,x_5))}_{\text{length } 2}$ of $\mathbb{R}^2 \times \mathbb{R}^3$ have length 2.

but elements $\underbrace{(x_1,x_2,x_3,x_4,x_5)}_{\text{length } 5}$ of $\mathbb{R}^5$ have length 5.

~~Proof~~: Define $T: \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$ by

$$T((x_1,x_2),(x_3,x_4,x_5)) = (x_1,x_2,x_3,x_4,x_5)$$

First, we will show that T is injective.

Let $((x_1,x_2),(x_3,x_4,x_5)) \in \text{null } T$, which means

$$T((x_1,x_2),(x_3,x_4,x_5)) = (0,0,0,0,0).$$

Then we have

$$(0,0,0,0,0) = T((x_1,x_2),(x_3,x_4,x_5))$$
$$= (x_1,x_2,x_3,x_4,x_5)$$

So

$$x_1=0, x_2=0, x_3=0, x_4=0, x_5=0$$

This means

$$((x_1,x_2),(x_3,x_4,x_5)) = ((0,0),(0,0,0)).$$

So $\text{null } T \subset \{((0,0),(0,0,0))\}$

$$T((0,0),(0,0,0)) = (0,0,0,0,0)$$

We also have

$$\{((0,0),(0,0,0))\} \subset \text{null } T.$$

Therefore, $\text{null } T = \{((0,0),(0,0,0))\}$

By 3.16 of Axler, T is injective.

Next, we will show that T is surjective.

For all $(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5$, we have

$$(x_1,x_2,x_3,x_4,x_5) = T((x_1,x_2),(x_3,x_4,x_5))$$
$$\in \text{range } T.$$

So $\mathbb{R}^5 \subset$ range T. But range T is a subspace of $\mathbb{R}^5$. So we have

$$\text{range } T = \mathbb{R}^5$$

So T is surjective.

Therefore, by 3.56 of Axler, T is invertible.

Next, we will show that T is linear.

- <u>Additivity</u>: For all $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}$, we have

$$T\Big(\big((x_1,x_2),(x_3,x_4,x_5)\big) + \big((y_1,y_2),(y_3,y_4,y_5)\big)\Big)$$
$$= T\Big(\big((x_1+y_1, x_2+y_2),(x_3+y_3, x_4+y_4, x_5+y_5)\big)\Big)$$
$$= (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5)$$
$$= (x_1,x_2,x_3,x_4,x_5) + (y_1,y_2,y_3,y_4,y_5)$$
$$= T\Big(\big((x_1,x_2),(x_3,x_4,x_5)\big)\Big) + T\Big(\big((y_1,y_2),(y_3,y_4,y_5)\big)\Big).$$

- <u>Homogeneity</u>: For all $\lambda \in \mathbb{F}$ and for all $x_1, x_2, x_3, x_4, x_5 \in \mathbb{F}$

we have $T\Big(\lambda\big((x_1,x_2),(x_3,x_4,x_5)\big)\Big) = T\Big(\big((\lambda x_1, \lambda x_2),(\lambda x_3, \lambda x_4, \lambda x_5)\big)\Big)$
$$= (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5)$$
$$= \lambda(x_1,x_2,x_3,x_4,x_5)$$
$$= \lambda T\big((x_1,x_2),(x_3,x_4,x_5)\big)$$

Therefore T is linear.

So T is invertible and linear.
Therefore, T is an isomorphism $\boxtimes$

3.75 Example

Find a basis of $\mathcal{P}_2(\mathbb{R}) \times \mathbb{R}^2$

Solution: $(1, (0,0)), (x, (0,0)), (x^2, (0,0)), (0, (1,0)), (0, (0,1))$

$1, x, x^2$ is a basis of $\mathcal{P}_2(\mathbb{R})$ $\quad$ $(1,0), (0,1)$ is a basis of $\mathbb{R}^2$

3.76 Dimension of a product is the sum of dimensions

Suppose $V_1, \dots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \dots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m.$$

Proof: Axler

Choose a basis of each V_j. For each basis vector of each V_j, consider the element of $V_1 \times \dots \times V_m$ that equals the basis vector in the j^{th} slot and 0 in the other slots. The list of all such vectors is linearly independent and spans $V_1 \times \dots \times V_m$. Therefore, it is a basis of $V_1 \times \dots \times V_m$, with length

$$\dim V_1 + \dots + \dim V_m \quad \boxtimes$$

My interpretation:

Let $j=1,\dots,m$. Let $v_{j,1},\dots,v_{j,n_j}$ be a basis of each V_j. Then $n_j = \dim V_j$, and the ith basis vector of V_j is v_{j,i_j} for $i=1,\dots,n_j$. So we have

$(v_{1,1},0,\dots,0),\dots,(0,\dots,0,v_{1,n_1})$ length $n_1 = \dim V_1$

$(v_{2,1},0,\dots,0),\dots,(0,\dots,0,v_{2,n_2})$ length $n_2 = \dim V_2$

$\vdots$

$(v_{m,1},0,\dots,0),\dots,(0,\dots,0,v_{m,n_m})$ length: $n_m = \dim V_m$

is a basis of $V_1 \times \dots \times V_m$

Total length: $n_1+\dots+n_m = \dim V_1 + \dots + \dim V_m$.

Products and Direct Sums

3.77 Products and Direct sums

Suppose that $U_1,\dots,U_m$ are subspaces of V. Define a linear map

$$\Gamma : U_1 \times \dots \times U_m \to U_1 + \dots + U_m \text{ by}$$
$$\Gamma(u_1,\dots,u_m) = u_1 + \dots + u_m.$$

Then $U_1 + \dots + U_m$ is a direct sum iff Γ is injective.

Proof Forward direction: If $U_1+\dots+U_m$ is a direct sum, then Γ is injective.

Suppose $(u_1,\dots,u_m) \in \operatorname{null}\Gamma$, so that

$$\Gamma(u_1,\dots,u_m) = \underbrace{(0+\dots+0)}_{m}$$

Since $U_1+\dots+U_m$ is a direct sum, by 1.44 of Axler, the only way to write the zero vector $0+\dots+0$ is to take $u_1=0,\dots,u_m=0$.

So $(u_1,\dots,u_m)=0$, and so null $\Gamma=\{0\}$. By 3.16 of Axler, Γ is injective.

Backward direction: If Γ is injective, then $U_1+\dots+U_m$ is a direct sum.

Since Γ is injective, by 3.16 of Axler, we have

$$\text{null } \Gamma = \{(\underbrace{0}_{u_1},\dots,\underbrace{0}_{u_m})\}$$

So the only way to write $0+\dots+0$ is to take

$$u_1=0,\dots,u_m=0.$$

By 1.44 of Axler, $U_1+\dots+U_m$ is a direct sum $\boxtimes$

3.78 A sum is a direct sum if and only if dimensions add up

Suppose V is finite-dimensional and $U_1,\dots,U_m$ are subspaces of V. Then $U_1+\dots+U_m$ is a direct sum iff

$$\dim(U_1+\dots+U_m)=\dim U_1+\dots+\dim U_m.$$

Proof: By the proof 3.77 of Axler, the map

$$\Gamma: U_1\times\dots\times U_m \to U_1+\dots+U_m \text{ defined by}$$

$$\Gamma(u_1,\dots,u_m)=u_1+\dots+u_m$$

is surjective. So the Fund. Thm of Linear Maps (3.22 of Axler) gives us

$\dim(U_1+\dots+U_m) = \dim \operatorname{range} \Gamma$ (b/c Γ is surjective, range $\Gamma = U_1+\dots+U_m$)

$= \dim(U_1\times\dots\times U_m) - \dim \operatorname{null} \Gamma$

by Fund. Thm of Linear Maps

$= \dim(U_1\times\dots\times U_m) - \dim\{0\}$

iff Γ is injective (3.16 of Axler)

$= \dim U_1\times\dots\times U_m)$

iff Γ is injective.

Combine w/ 3.77 & 3.76 of Axler to conclude that $U_1+\dots+U_m$ is a direct sum iff we have

$\dim(U_1+\dots+U_m) = \dim(U_1\times\dots\times U_m)$

$= \dim U_1 + \dots + \dim U_m$ by 3.76

$\blacksquare$

1/22 3.E continued

Quotients of Vector Spaces

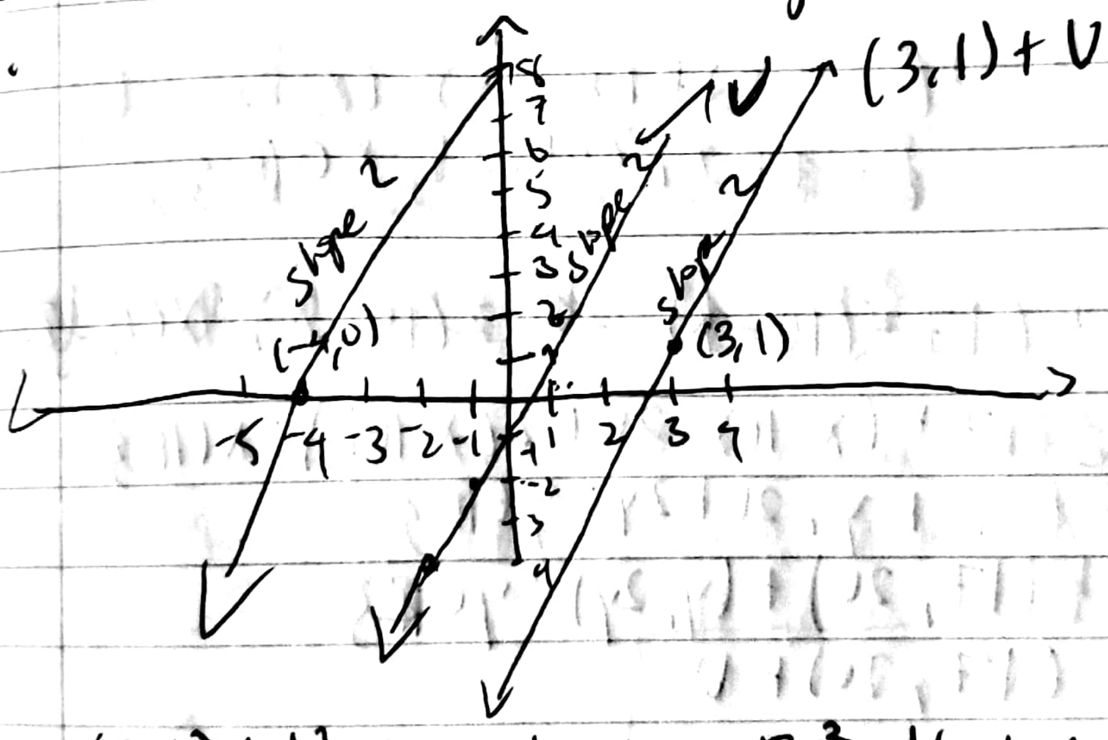
3.79 Definition

Suppose $v \in V$ and U is a subspace of V. Then $v+U$ is the subset of V defined

$$v+U=\{v+u : u\in U\}.$$

3.80 Example

Let $V=\mathbb{R}^2$ and $U=\{(x,2x)\in\mathbb{R}^2 : x\in\mathbb{R}\}$.
Then U is the line in $\mathbb{R}^2$ through the origin w/ slope 2.



So $(3,1)+U$ is a line in $\mathbb{R}^2$ that contains the point $(3,1)$ and has slope 2. and $(-4,0)+U$ is a line in $\mathbb{R}^2$ that contains the point $(-4,0)$ and has slope 2.

$$(3,1)+U=\{(3,1)+(x,2x) : x\in\mathbb{R}\}$$
$$=\{(3+x,1+2x) : x\in\mathbb{R}\}$$

$(-4,0)+U = \{(-4,0)+(x,2x) : x \in \mathbb{R}\}$

$= \{(-4+x,\ 2x) : x \in \mathbb{R}\}$.

Prove: Since ~~(8,14)~~ (7,0) and (17,20) lie on same line, ~~(8,14)~~ (7,0) + U = (17,20) + U.

Proof: $\boxed{\cancel{(8,14)}\,(7,0)+U} = \{\cancel{(8,14)}\,(7,0)+(x,2x) : (x,2x) \in U\}$

$= \{\cancel{(8,14)}\,(7,0)+(x,2x) : x \in \mathbb{R}\}$

$= \{(\underset{7}{\cancel{8}}+x,\ \underset{0}{\cancel{14}}+2x) : x \in \mathbb{R}\}$

$= \{(17-10+x,\ 20-20+2x) : x \in \mathbb{R}\}$

$(17,20)+U = \{(17,20)+(x,2x) : (x,2x) \in U\}$

$= \{(17+x,\ 20+2x) : x \in \mathbb{R}\}$

$\cancel{= \{(17-(10-x),\ 20-2(10-x)) : x \in \mathbb{R}\}}$

Let $y = x-10$, since $x \in \mathbb{R}$, it follows that $y \in \mathbb{R}$

$= \{(17+(x-10),\ 20+2(x-10)) : x \in \mathbb{R}\}$

$= \{(17+y,\ 20+2y) : y \in \mathbb{R}\}$

$= \{(17,20)+(y,2y) : y \in \mathbb{R}\}$

$\boxed{= (17,20)+U}$

3.81 Definition

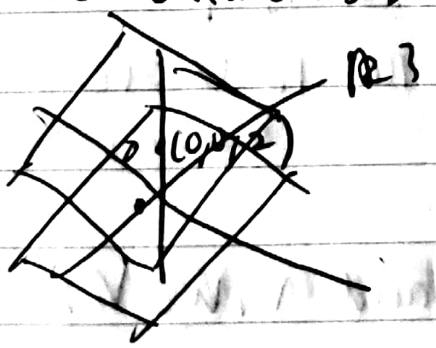
- An <u>affine subset</u> of V is a subset of V of the form $v+U$ for some $v \in V$ and some subspace U of V.
- If U is a subspace of V, for all $v \in V$, the affine subset $v+U$ is said to be <u>parallel</u> to U.

3.82 Example

- Let $V=\mathbb{R}^2$ and $U=\{(x,2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, as in Example 3.80. Then all the lines in $\mathbb{R}^2$ with slope 2 are parallel to U. And these lines are affine subsets in $\mathbb{R}^2$.
- Let $V=\mathbb{R}^3$ and $U=\{(x,y,0) \in \mathbb{R}^3 : x,y \in \mathbb{R}\}$
Then the affine subsets of $\mathbb{R}^3$ are all the planes in $\mathbb{R}^3$ that are parallel to U. For example,

$$(0,0,2)+U=\{(0,0,2)+(x,y,0) : x,y \in \mathbb{R}\}$$
$$=\{(x,y,2) : x,y \in \mathbb{R}\}$$

is an affine subset of $\mathbb{R}^3$ and is parallel to U.



3.83 Definition:

Let U be a subspace of V. Then the <u>quotient space</u> V/U is the set of all affine subsets of V parallel to U, written:

quotient space $V/U = \{v + U : v \in V\}$

Ex: $(7,0) + U$ is an affine subset of $\mathbb{R}^2$.

$(7,0) + U \in \mathbb{R}^2/U$.

3.84 Example

- If $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, then $\mathbb{R}^2/U$ is the set of all lines in $\mathbb{R}^2$ that have slope 2.
- If U is a line in $\mathbb{R}^3$ containing the origin, then $\mathbb{R}^3/U$ is the set of all lines in $\mathbb{R}^3$ parallel to U.

for ex:, $U_1 = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$.

$\mathbb{R}^3/U_1 = \{(0, 0, z) + U_1 : x, y, z \in \mathbb{R}\}$

$U_2 = \{(0, y, z) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$

$\mathbb{R}^3/U_2 = \{(x, 0, 0) + U_2 : x, y, z \in \mathbb{R}\}$

Important result for upcoming exams

3.85 Two affine subsets parallel to U are equal or disjoint

Let U be a subspace of V and $v, w \in V$.
Then the following are equivalent:

(a) $v - w \in U$

(b) $v + U = w + U$

(c) $(v + U) \cap (w + U) \neq \emptyset$

Proof: (a) implies (b):

Suppose (a) holds: $v-w \in U$. Let $u \in U$ be arbitrary.

Since U is a subspace of V, in particular it is closed under addition. Since $u \in U$ and $v-w \in U$, we have $(v-w)+u \in U$. For all $u \in U$, we have

$$\begin{aligned} v+u &= w+v-w+u \\ &= w+\underbrace{[(v-w)+u]}_{\in U} \\ &\in w+U. \end{aligned}$$

~~Similarly for all $u \in U$~~

Therefore, $v+U \subset w+U$.

Similarly, for all $u \in U$, we have

$$\begin{aligned} w+u &= v+w-v+u \\ &= v+\underbrace{[-(v-w)+u]}_{\in U} \\ &\in v+U. \end{aligned}$$

Therefore, $w+U \subset v+U$.

So we conclude the set equality $v+U = w+U$, which is (b).

(b) implies (c)

Suppose (b) holds: $v+U = w+U$. Then there exists $u \in U$ that satisfies

$$\begin{aligned} v+u &\in v+U \\ &= w+U. \end{aligned}$$

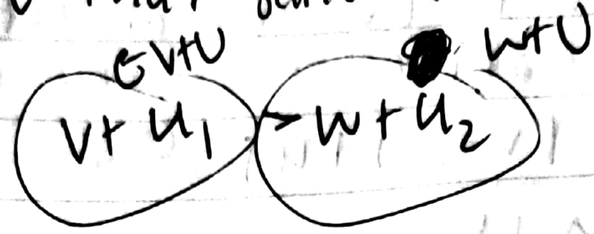
So $v+u \in v+U$ and $v+u \in w+U$. That is,

$$v+u \in (v+U) \cap (w+U).$$

In other words, $(v+U) \cap (w+U) \neq \emptyset$, which is (c).

(c) implies (a)

Suppose (c) holds: $(v+U) \cap (w+U) \neq \emptyset$. Then there exist $u_1, u_2 \in U$ that satisfies



Since U is a subspace of V, it is closed under addition and scalar multiplication, which means $u_1 - u_2 \in U$. In fact, we have

$$\begin{aligned} v - w &= u_2 - u_1 \\ &= -(u_1 - u_2) \\ &\in U, \end{aligned}$$

■

3.86 Definition

Let U be a subspace of V. Then:

- **addition** is defined on V/U by
$$(v+U)+(w+U) = (v+w)+U$$
- **scalar multiplication** is defined on V/U by
$$\lambda(v+U) = (\lambda v)+U.$$

3.87 Quotient space is a vector space

Let U be a subspace of V. Then V/U is a vector space with respect to the operations defined in Definition 3.86.

Proof: Let $v, w \in V$ be arbitrary.

First, we need to show that the operations of addition and scalar multiplication make sense on V/U.

Suppose $\hat{v}, \hat{w} \in V$ satisfy $v + U = \hat{v} + U$ and $w + U = \hat{w} + U$.

First, we will show that addition makes sense on V/U.

Since U is a subspace of V, it is closed under addition. So

$$(v+w) - (\hat{v}+\hat{w}) = v - \hat{v} + w - \hat{w} \in U.$$

By 3.85 of Axler,

$$(v+w) + U = (\hat{v}+\hat{w}) + U.$$

So addition makes sense on V/U.

Now let $\lambda \in \mathbb{F}$ be arbitrary. Suppose $\hat{v} \in V$ satisfies $v + U = \hat{v} + U$.

By 3.85 of Axler, $v - \hat{v} \in U$. Since U is a subspace of V, it is closed under scalar multiplication, which means $\lambda(u - \hat{u}) \in U$. So we have

$$\lambda v - \lambda \hat{v} = \lambda(v - \hat{v}) \in U.$$

By 3.85 of Axler,

$$\lambda v + U = \lambda \hat{v} + U$$

So scalar multiplication makes sense on V/U.

Next, we will show that V/U satisfies all axioms of a vector space. Let $v, w, x \in V$ and $a, b \in \mathbb{F}$ be arbitrary.

- Commutativity: $$(v+U)+(w+U) = (v+w)+U$$ $$= (w+v)+U$$ $$= (w+U)+(v+U).$$

- Associativity: $$((v+U)+(w+U))+(x+U) = ((v+w)+U)+(x+U)$$ $$= ((v+w)+x)+U$$ $$= (v+(w+x))+U$$ $$= (v+U)+((w+x)+U)$$ $$= (v+U)+((w+U)+(x+U)).$$

- Additive Identity: $$(v+U)+(0+U) = (v+0)+U$$ $$= v+U$$

- Additive Inverse: $$(v+U)+((-v)+U) = (v+(-v))+U$$ $$= 0+U$$

- Multiplicative Identity: $$1(v+U) = (1v)+U$$ $$= v+U$$

- Distributive properties:

$$a((v+U)+(w+U)) = a((v+w)+U)$$ $$= a(v+w)+U$$ $$= (av+aw)+U$$ $$= ((av)+U)+((aw)+U)$$ $$= a(v+U)+a(w+U).$$

and

$$(a+b)(v+U) = ((a+b)v)+U$$ $$= (av+bv)+U$$ $$= ((av)+U)+((bv)+U)$$ $$= a(v+U)+b(v+U)$$

$\blacksquare$

3.88 Definition

Let U be a subspace of V. The quotient map is the linear map
$$\pi : V \to V/U$$ defined by
$$\pi(v) = v + U \quad \text{for all } v \in V.$$

3.89 Dimension of a quotient space

Suppose V is finite-dimensional and U is a subspace of V. Then
$$\dim V/U = \dim V - \dim U.$$

Proof: Let $\pi : V \to V/U$ be the quotient map.

First, we claim $\operatorname{null} \pi = U$.

Since $v \in U$, we have $v - 0 = v \in U$, so by 3.85 of Axler,
$$v + U = 0 + U.$$

In fact, we have
$$\pi(v) = v + U$$
$$= 0 + U.$$

So $v \in \operatorname{null} \pi$, and so $U \subset \operatorname{null} \pi$.

If $v \in \operatorname{null} \pi$, then $\pi(v) = 0 + U$.

Since we also have $\pi(v) = v + U$, we conclude
$$v + U = 0 + U.$$

By 3.85 of Axler,
$$v = v - 0 \in U.$$

So $\operatorname{null} \pi \subset U$.

Therefore, we conclude the set equality
$$\operatorname{null} \pi = U.$$

Next claim: range $\pi = V/U$.

Let $w \in$ range π

Then $w = \pi(v)$ for some $v \in U$.

In fact, by Definition 3.88,

we have

$$\begin{aligned} w &= \pi(v) \\ &= v + U \\ &\in V/U \end{aligned}$$

So we get range $\pi \subset V/U$

Suppose we have

$v + U \in V/U$

By Definition 3.88,

$$\begin{aligned} v + U &= \pi(v) \\ &\in \text{range } \pi \end{aligned}$$

So $V/U \subset$ range π

Therefore, range $\pi = V/U$

By the Fundamental Thm of Lin. Maps (3.22 of Axler) we have

$$\begin{aligned} \dim V &= \dim \text{null } \pi + \dim \text{range } \pi \\ &= \dim U + \dim V/U \end{aligned}$$

as desired. $\blacksquare$

/23 3.90 Definition:

Suppose $T \in \mathcal{L}(V,W)$. Define $\tilde{T}: V/(\text{null } T) \to W$ by
$$\tilde{T}(v + \text{null } T) = Tv.$$

Show that $\tilde{T}$ makes sense ($\tilde{T}$ is well-defined)

Suppose $u, v \in V$ satisfy
$$u + \text{null } T = v + \text{null } T$$

By 3.85 of Axler, we have
$$u - v \in \text{null } T$$

This means
$$T(u-v) = 0$$

In fact, we have
$$Tu - Tv = T(u-v) = 0,$$

so $Tu - Tv$

Therefore,
$$\begin{aligned}\tilde{T}(u + \text{null } T) &= Tu \\ &= Tv \\ &= \tilde{T}(v + \text{null } T)\end{aligned}$$

and so $\tilde{T}$ is well-defined

3.91 Null space and range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V,W)$. Then:

(a) $\tilde{T} : V/(\text{null } T) \to W$ is a linear map;
$\tilde{T} \in \mathcal{L}(V/(\text{null } T), W)$

(b) $\tilde{T}$ is injective

(c) range $\tilde{T}$ = range T

(d) $V/(\text{null } T)$ is isomorphic to range T

Proof: (a) Let $u, v \in V$ and $\lambda \in \mathbb{F}$

• Additivity: $\tilde{T}((u + \text{null } T) + (v + \text{null } T))$

$$= \tilde{T}((u+v) + \text{null } T)$$
$$= T(u+v)$$
$$= Tu + Tv$$
$$= \tilde{T}(u + \text{null } T) + \tilde{T}(v + \text{null } T)$$

• Homogeneity:

$$\tilde{T}(\lambda(v + \text{null } T)) = \tilde{T}((\lambda v) + \text{null } T)$$
$$= T(\lambda v)$$
$$= \lambda T v$$
$$= \lambda \tilde{T}(v + \text{null } T).$$

Therefore, $\tilde{T}$ is linear.

(b): Suppose $v \in V$ ~~and~~ satisfies $\tilde{T}(v + \text{null } T) = 0$

Then we have

$$Tv = \tilde{T}(v + \text{null } T)$$
$$= 0.$$

~~By 3.85 of Axler~~, So $v - 0 = v \in \text{null } T$

By 3.85 of Axler,

$v + \text{null } T = 0 + \text{null } T$ — Additive identity of $V/(\text{null } T)$

~~So $\tilde{T}(0 + \text{null } T) = \tilde{T}(v + \text{null } T)$~~ ~~[illegible]~~

~~By part (a), $\tilde{T}$ is linear. So 3.11 of Axler, we have~~

~~$\tilde{T}(0 + \text{null } T) = 0$~~

~~So $0 + \text{null } T \in \text{null } \tilde{T}$, or $\{0 + \text{null } T\} \subset \text{null } \tilde{T}$~~

Therefore, $\text{null } \tilde{T} \subset \{0 + \text{null } T\}$.

But $\tilde{T}(0 + \text{null } T) = 0$ since $\tilde{T}$ is linear

So $\{0 + \text{null } T\} \subset \text{null } \tilde{T}$

So $\text{null } \tilde{T} = \{0 + \text{null } T\}$

By 3.16 of Axler,

$\tilde{T}$ is injective.

(c): For all $v \in V$, $\tilde{T}(v + \text{null } T) = Tv$.
Suppose $w \in \text{range } T$. Then $w = Tv$ for some $v \in V$. In fact,

$$w = Tv$$
$$= \tilde{T}(v + \text{null } T)$$
$$\in \text{range } \tilde{T}$$

So $\text{range } T \subset \text{range } \tilde{T}$.

Suppose $x \in \text{range } \tilde{T}$. Then $x = \tilde{T}(v + \text{null } T)$ for some $v \in V$.
In fact,

$$x = \tilde{T}(v + \text{null } T)$$
$$= Tv$$
$$\in \text{range } T$$

So $\text{range } \tilde{T} \subset \text{range } T$.
Therefore, we conclude $\text{range } \tilde{T} = \text{range } T$.

(d): By part (c), $\text{range } \tilde{T} = \text{range } T$.
If we think of $\tilde{T}$ as a map into $\text{range } \tilde{T}$,
$\tilde{T} : V/(\text{null } T) \to \text{range } T$ is surjective

So $\tilde{T}$ is also surjective.

By part (b), $\tilde{T}$ is also injective.
Therefore by 3.56 of Axler, $\tilde{T}$ is invertible
By part (a), $\tilde{T}$ is linear

Therefore, $\tilde{T}: V/(\text{null } T) \to \text{range } T$ is an isomorphism