7/17/2019 ## 3E Products and Quotients of Vector Spaces

Defn 3.71 Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$.

- The product $V_1 \times \cdots \times V_m$ is defined by
$$V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}$$
- Addition on $V_1 \times \cdots \times V_m$ is defined by
$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m)$$
- Scalar Multiplication on $V_1 \times \cdots \times V_m$ is defined by
$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m)$$

Eg 3.72

- $\underbrace{((5 - 6x + 4x^2), (3, 8, 7))}_{\text{length 2}} \in P_2(\mathbb{R}) \times \mathbb{R}^3$
- $\underbrace{((1, 2), (3, 4, 5))}_{\text{length 2}} \in \mathbb{R}^2 \times \mathbb{R}^3$
- $\underbrace{((1), (2, 3), (4, 5))}_{\text{length 3}} \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^3$

Extn 3.73 Product of vector spaces is a vector space

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

proof: Let $u_i, v_i, w_i \in V_i$ for each $i = 1, \dots, m$ and let $\lambda \in \mathbb{F}$.

- Commutativity: Since each V_i is a vector space, we have $u_i + v_i = v_i + u_i$. So we have
$$\begin{aligned}(u_1, \dots, u_m) + (v_1, \dots, v_m) &= (u_1 + v_1, \dots, u_m + v_m) \\ &= (v_1 + u_1, \dots, v_m + u_m) \\ &= (v_1, \dots, v_m) + (u_1, \dots, u_m)\end{aligned}$$
- associativity: Since V_i is a vector space, we have $(u_i + v_i) + w_i = u_i + (v_i + w_i)$. So we have
$$\begin{aligned}((u_1, \dots, u_m) + (v_1, \dots, v_m)) + (w_1, \dots, w_m) &= \\ &= (u_1 + v_1, \dots, u_m + v_m) + (w_1, \dots, w_m) \\ &= ((u_1 + v_1) + w_1, \dots, (u_m + v_m) + w_m) \\ &= (u_1 + (v_1 + w_1), \dots, (u_m + (v_m + w_m)) \\ &= (u_1, \dots, u_m) + (v_1 + w_1, \dots, v_m + w_m) \\ &= (u_1, \dots, u_m) + ((v_1, \dots, v_m) + (w_1, \dots, w_m))\end{aligned}$$

- additive identity: We have $\overbrace{(0, \cdots, 0)}^{\text{zero vectors}} \in V_1 \times \cdots \times V_m$.
  And we have $(v_1, \cdots, v_m) + (0, \cdots, 0) = (v_1 + 0, \cdots, + v_m + 0)$
  $= (v_1, \cdots, v_m)$

  So $(0, \cdots, 0)$ is the additive identity of $V_1 \times \cdots \times V_m$.
- additive inverse: We have $(-v_1, \cdots, -v_n) \in V_1 \times \cdots \times V_m$.
  And it satisfies
  $$(v_1, \cdots, v_m) + (-v_1, \cdots, -v_m) = (v_1 + (-v_1), \cdots, (v_m) + (-v_m))$$
  $$= (v_1 - v_1, \cdots, v_m - v_m)$$
  $$= (0, \cdots, 0)$$
  So $(-v_1, \cdots, -v_m)$ is the additive inverse of $V_1 \times \cdots \times V_m$.
- Multiplication Identity: We have
  $$1(v_1, \cdots, v_m) = (1v_1, \cdots, 1v_m)$$
  $$= (v_1, \cdots, v_m)$$
- Distributive properties:
  For all $a, b \in \mathbb{F}$, we have
  $$a((u_1, \cdots, u_m) + (v_1, \cdots, v_m)) = a(u_1 + v_1, \cdots, u_m + v_m)$$
  $$= (a(u_1 + v_1), \cdots, a(u_m + v_m))$$
  $$= (au_1 + av_1, \cdots, au_m + av_m)$$
  $$= (au_1, \cdots, au_m) + (av_1, \cdots, av_m)$$
  $$= a(u_1, \cdots, u_m) + a(v_1, \cdots v_m)$$
  and $(a+b)(v_1, \cdots, v_m) = ((a+b)v_1, \cdots, (a+b)v_m)$
  $$= (av_1 + bv_1, \cdots, av_m + bv_m)$$
  $$= (av_1, \cdots, av_m) + (bv_1, \cdots, bv_m)$$
  $$= a(v_1, \cdots, v_m) + b(v_1, \cdots, v_m)$$

Eg 3.74 Show that $\mathbb{R}^2 \times \mathbb{R}^3$ is isomorphic to $\mathbb{R}^5$

Note that, as vector spaces, $\mathbb{R}^2 \times \mathbb{R}^3 \neq \mathbb{R}^5$ because elements $\underbrace{(\underbrace{(x_1, x_2)}_{\text{length 2}}, \underbrace{(x_3, x_4, x_5)}_{\text{length 3}})}_{\text{length 2}}$ of $\mathbb{R}^2 \times \mathbb{R}^3$ has length 2 but elements $\underbrace{(\underbrace{x_1}_{1}, \underbrace{x_2}_{1}, \underbrace{x_3}_{1}, \underbrace{x_4}_{1}, \underbrace{x_5}_{1})}_{\text{length 5}}$ of $\mathbb{R}^5$ has length 5.

proof: Define $T: \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$ by $T((x_1, x_2), (x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4, x_5)$.

① First, we will show that T is injective. Let $((x_1, x_2), (x_3, x_4, x_5)) \in \operatorname{null} T$, which means
$$T((x_1, x_2), (x_3, x_4, x_5)) = (0, 0, 0, 0, 0)$$
Then we have $(0,0,0,0,0) = T((x_1, x_2), (x_3, x_4, x_5))$
$$= (x_1, x_2, x_3, x_4, x_5)$$
So, $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0$

This means $((x_1, x_2), (x_3, x_4, x_5)) = ((0,0), (0,0,0))$

So $\operatorname{null} T \subset \{((0,0),(0,0,0))\}$
$$T((0,0), (0,0,0)) = (0,0,0,0,0)$$
we also have $\{((0,0), (0,0,0))\} \subset \operatorname{null} T$.

Therefore, $\operatorname{null} T = \{((0,0), (0,0,0))\}$

By 3.16 of Axler, T is injective.

② Next, we will show that T is surjective.

For all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, we have
$$(x_1, x_2, x_3, x_4, x_5) = T((x_1, x_2), (x_3, x_4, x_5)) \in \operatorname{range} T.$$
So $\mathbb{R}^5 \subset \operatorname{range} T$. But $\operatorname{range} T$ is a subspace of $\mathbb{R}^5$.

So we have $\operatorname{range} T = \mathbb{R}^5$.

So T is surjective.

Therefore, by 3.56 of Axler, T is invertible.

③ Next, we will show that T is linear.
- Additivity: For all $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}$, we have
  $$T((x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4, y_5))$$
  $$= T((x_1 + y_1, x_2 + y_2), (x_3 + y_3, x_4 + y_4, x_5 + y_5))$$
  $$= (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + y_5)$$
  $$= (x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)$$
  $$= T((x_1, x_2), (x_3, x_4, x_5)) + T((y_1, y_2), (y_3, y_4, y_5))$$
- Homogeneity: For all $\lambda \in \mathbb{F}$ and for all $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$, we have

$$T(\lambda((x_1,x_2),(x_3,x_4,x_5))) = T((\lambda x_1, \lambda x_2),(\lambda x_3, \lambda x_4, \lambda x_5))$$
$$= (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5)$$
$$= \lambda(x_1, x_2, x_3, x_4, x_5)$$
$$= \lambda T((x_1,x_2),(x_3,x_4,x_5))$$

Therefore T is linear.
So T is invertible and linear.
Therefore, T is an isomorphism. $\square$

Eg 3.75 Find a basis of $P_2(\mathbb{R}) \times \mathbb{R}^2$

Soln: $(1,(0,0)), (x,(0,0)), (x^2,(0,0)), (0,(1,0)), (0,(0,1))$

$1, x, x^2$ is a basis of $P_2(\mathbb{R})$

$(1,0), (0,1)$ is a basis of $\mathbb{R}^2$

Defn 3.76 Dimension of a product is the sum of dimensions.

Suppose $V_1, \dots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite dimensional and
$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m.$$

proof: AXLER Choose a basis of each V_j. For each basis vector of each V_j consider the element of $V_1 \times \cdots \times V_m$ that equals the basis vector in the j^{th} slot and 0 in the other slots. The list of all such vectors is linearly independent and spans $V_1 \times \cdots \times V_m$. Therefore, it is of $V_1 \times \cdots \times V_m$, with length
$$\dim V_1 + \cdots + \dim V_m. \quad \square$$

Ryan's Interpretation:
Let $j = 1, \dots, m$. Let $v_{j,1}, \dots, v_{j,n_j}$ be a basis of each V_j. Then $n_j = \dim V_j$, and the i^{th} basis vector of V_j is v_{j,i_j} for $i = 1, \dots, n_j$. So we have ?

$(v_{1,1}, 0, \dots, 0), \dots, (0, \dots, 0, v_{1,n_1})$, length: $n_1 = \dim V_1$

$(v_{2,1}, 0, \dots, 0), \dots, (0, \dots, 0, v_{2,n_2})$, length: $n_2 = \dim V_2$

$\vdots$

$(v_{n,1}, 0, \dots, 0), \dots, (0, \dots, 0, v_{m,n_m})$, length: $n_m = \dim V_m$

Total length:
$n_1 + n_2 + \cdots + n_m = \dim V_1 + \dim V_2 + \cdots + \dim V_m$

is a basis of $V_1 \times \cdots \times V_m$.

Products and Direct Sums

Defn 3.77 Products and Direct Sums

Suppose that $U_1, \dots, U_m$ are subspaces of V. Define a linear map
$$\Gamma: U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m \text{ by}$$
$$\Gamma(u_1, \dots, u_m) = u_1 + \cdots + u_m.$$
Then $U_1 + \cdots + U_m$ is a direct sum if and only if Γ is injective.

proof: Forward Direction: If $U_1 + \cdots + U_m$ is a direct sum, then Γ is injective. Suppose $(u_1, \dots, u_m) \in \operatorname{null}\Gamma$, so that
$$T(u_1, \dots, u_m) = \underbrace{0 + \cdots + 0}_{m}.$$

Since $U_1 + \cdots + U_m$ is a direct-sum, by 1.44 of Axler, the only way to write the zero vector $0 + \cdots + 0$ is to take $u_1 = 0, \dots, u_m = 0$.
So $(u_1, \dots, u_m) = 0$, and so $\operatorname{null}\Gamma = \{0\}$. By 3.16 of Axler, Γ is injective.

Backward Direction: If Γ is injective, then $U_1 + \cdots + U_m$ is a direct sum. Since Γ is injective, by 3.16 of Axler, we have $\operatorname{null}\Gamma = \{(\overset{u_1}{0}, \dots, \overset{u_m}{0})\}$. So the only way to write $0 + \cdots + 0$ is to take
$$u_1 = 0, \dots, u_m = 0.$$
By 1.44 of Axler, $U_1 + \cdots + U_m$ is a direct sum. $\square$

Defn 3.78 A sum is a direct sum if and only if dimensions add up.

Suppose V is finite-dimensional and $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if
$$\dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m.$$

proof: By the proof of 3.77 of Axler, the map
$$\Gamma: U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m \text{ defined by}$$

$\Gamma(U_1, \dots, U_m) = u_1 + \dots + u_m$ is surjective.

So the Fundamental Theorum of Linear Maps (3.22 of Axler) gives us

$$\dim(U_1 + \dots + U_m) = \dim \operatorname{range} \Gamma \leftarrow \left(\text{because } \Gamma \text{ is surjective, range } \Gamma = U_1 + \dots + U_m\right)$$
$$= \dim(U_1 \times \dots \times U_m) - \dim \operatorname{null} \Gamma \leftarrow \text{by Fund. Thm of Linear Maps}$$
$$= \dim(U_1 \times \dots \times U_m) - \dim\{0\} \leftarrow \text{if and only if } \Gamma \text{ is injective (3.16 of Axler)}$$
$$= \dim(U_1 \times \dots \times U_m)$$

if and only if Γ is injective.

Combine with 3.77 and 3.76 of Axler to conclude that $U_1 + \dots + U_m$ is a direct sum if and only if we have

$$\dim(U_1 + \dots + U_m) = \dim(U_1 \times \dots \times U_m)$$
$$= \dim U_1 + \dots + \dim U_m \quad \text{by 3.76 of Axler}$$

☐

$\boxed{\text{3E Cont.}}$

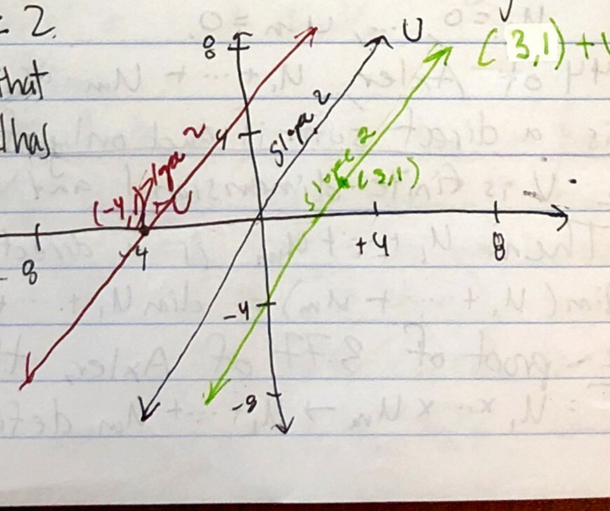
Quotients of Vector Spaces

Defn 3.79 Suppose $v \in V$ and U is a subspace of V. Then $v + U$ is the subset of V defined

$$v + U = \{v + u : u \in U\}$$

Eg 3.80 Let $V = \mathbb{R}^2$ and $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Then U is the line in $\mathbb{R}^2$ through the origin with slope 2.

So $(3,1)+U$ is a line in $\mathbb{R}^2$ that contains the point $(3,1)$ and has slope 2 and $(-4,0)+U$ is a line in $\mathbb{R}^2$ that contains the point $(-4,0)$ and has slope 2

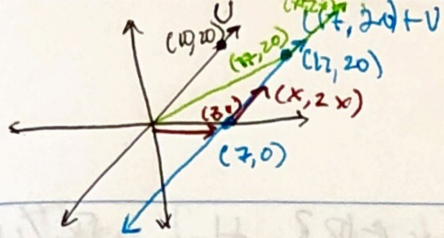


$(3,1)+U = \{(3,1)+(x,2x) : x \in \mathbb{R}\}$

$= \{(3+x, 1+2x) : x \in \mathbb{R}\}$

$(-4,0)+U = \{(-4,0)+(x,2x) : x \in \mathbb{R}\}$

$= \{(-4+x, 0+2x) : x \in \mathbb{R}\}$



Prove: Since $(7,0)$ and $(17,20)$ lie on the same line, $(7,0)+U = (17,20)+U$

proof: $(17,20)+U = \{(17,20)+(x,2x) : (x,2x) \in U\}$
$= \{(17+x, 20+2x) : x \in \mathbb{R}\}$

$\boxed{(7,0)+U} = \{(7,0)+(x,2x) : (x,2x) \in U\}$
$= \{(7+x, 2x) : x \in \mathbb{R}\}$
$= \{(17-10+x, 20-20+2x) : x \in \mathbb{R}\}$
$= \{(17+(x-10), 20+2(x-10)) : x \in \mathbb{R}\}$
$= \{(17+y, 20+2y) : y \in \mathbb{R}\}$
$= \{(17,20)+(y,2y) : y \in \mathbb{R}\}$
$= \boxed{(17,20)+U}$

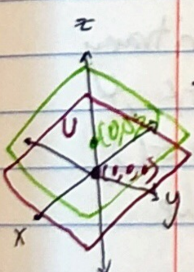
[Similarly
$(8,2)+U = (17,20)+U$
$(8,2)+U = (7,0)+U$
$(8,4)+U \neq (17,20)+U$]

let $y = x-10$ since $x \in \mathbb{R}$, it follows that $y \in \mathbb{R}$

Defn 3.81 · An affine subset of V is a subset of V of the form $v+U$ for some $v \in V$ and some subspace U of V.

· If U is a subspace of V, for all $v \in V$, the affine subset $v+U$ is said to be parallel to U.

Eg 3.82 · Let $V = \mathbb{R}^2$ and $U = \{(x,2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, as Eg 3.80. Then all the lines in $\mathbb{R}^2$ with slope 2 are parallel to U. And these lines are affine subsets in $\mathbb{R}^2$



· Let $V = \mathbb{R}^3$ and $U = \{(x,y,0) \in \mathbb{R}^3 : x,y \in \mathbb{R}\}$
Then the affine subsets of $\mathbb{R}^3$ are all planes in $\mathbb{R}^3$ that are parallel to U. for example,

$(0,0,2)+U = \{(0,0,2)+(x,y,0) : x,y \in \mathbb{R}\}$
$= \{(x,y,2) : x,y \in \mathbb{R}\}$

is an affine subset of $\mathbb{R}^3$ and parallel to U.

Defn 3.83 Let U be a subspace of V. Then the quotient space V/U is the set of all affine subsets of V parallel to U written:

quotient space $V/U = \{v+U : v \in V\}$

[eg $(7,0)+U$ is an affine subset of $\mathbb{R}^2$
$(7,0)+U \in \mathbb{R}^2/U$]

Eg 3.84 · If $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, then $\mathbb{R}^2/U$ is the set of all lines in $\mathbb{R}^2$ that have slope 2.

· If U is a line in $\mathbb{R}^3$ containing the origin, then $\mathbb{R}^3/U$ is the set of all lines in $\mathbb{R}^3$ parallel to U.

eg, $U_1 = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$

$\mathbb{R}^3/U_1 = \{(0,0,z) + U_1 : x, y, z \in \mathbb{R}\}$

$U_2 = \{(0, y, z) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$,

$\mathbb{R}^3/U_2 = \{(x, 0, 0) + U_2 : x, y, z \in \mathbb{R}\}$

* IMPORTANT RESULT FOR UPCOMING Exams!!! *

Thm 3.85 Two affine subsets parallel to U are equal or disjoint

∴ Let U be a subspace of V and $v, w \in V$. Then the following are equivalent:

(a) $v - w \in U$

(b) $v + U = w + U$

(c) $(v + U) \cap (w + U) \neq \emptyset$

proof: ① (a) implies (b):

Suppose (a) holds: $v - w \in U$. Let $u \in U$ be arbitrary. Since U is a subspace of V, in particular it is closed under addition. Since $u \in U$ and $v - w \in U$, we have $(v-w) + u \in U$. For all $u \in U$, we have

$$v + u = w + v - w + u \in U$$
$$= w + ((v - w) + u) \in w + U.$$

Therefore, $v + U \subset w + U$.

Similarly, for all $u \in U$, we have

$$w + u = v + w - v + u \in U$$
$$= v + (-(v - w) + u)$$
$$\in v + U$$

Therefore, $w + U \subset v + U$.

So we conclude the set equality

$$v + U = w + U,$$

which is (b).

② (b) implies (c):

Suppose (b) holds: $v + U = w + U$. Then there exists $u \in U$ that satisfies

$$v + u \in v + U$$
$$= w + U.$$

So $v + u \in v + U$ and $v + u \in w + U$.

That is, $v + u \in (v + U) \cap (w + U)$.

In other words,

$$(v + U) \cap (w + U) \neq \emptyset,$$

which is (c).

③ (c) implies (a):

Suppose (c) holds: $(v + U) \cap (w + U) \neq \emptyset$. Then there exist $u_1, u_2 \in U$ that satisfies

$$\underbrace{v + u_1}_{\in v+U} = \underbrace{w + u_2}_{\in w+U}$$

Since U is a subspace of V, it is closed under addion and scalar multiplication, which means $u_1 - u_2 \in U$. In fact, we have

$$v - w = u_2 - u_1$$
$$= -(u_1 - u_2)$$
$$\in U$$

which is (a)

$\blacksquare$

1 **Defn 3.86** Let U be a subspace of V. Then:

- addition is defined on V/U by
$$(v+U)+(w+U) = (v+w)+U.$$
- scalar multiplication is defined on V/U by
$$\lambda(v+U) = (\lambda v)+U.$$

Thm 3.87 Quotient Space is a vector space

Let U be a subspace of V. Then V/U is a vector space with respect to the operations defined in 3.86.

proof. Let $v, w \in V$ be arbitrary.

① First, we need to show that the operations of addition and scalar multiplication make sense on V/U.
Suppose $\hat{v}, \hat{w} \in V$ satisfy $v+U = \hat{v}+U$ and $w+U = \hat{w}+U$.

ⓐ First, we will show that addition makes sense on V/U
since U is a subspace of V, it is closed under addition. So $(v+w)-(\hat{v}+\hat{w}) = v-\hat{v} + w-\hat{w} \in U$.
By 3.85 of Axler,
$$(v+w)+U = (\hat{v}+\hat{w})+U.$$
So addition makes sense on V/U.

ⓑ Now let $\lambda \in \mathbb{F}$ be arbitrary. Suppose $\hat{v} \in V$ satisfies $v+U = \hat{v}+U$. By 3.85 of Axler, $v-\hat{v} \in U$.
Since U is a subspace of V, it is closed under scalar multiplication, which means $\lambda(u-\hat{u}) \in U$.
So we have
$$\lambda v - \lambda\hat{v} = \lambda(v-\hat{v}) \in U.$$
By 3.85 of Axler,
$$\lambda v+U = \lambda\hat{v}+U.$$
So scalar multiplication makes sense on V/U.

② Next, we will show that V/U satisfies all axioms of a vector space.
Let $v, w, x \in V$ and $\lambda \in \mathbb{F}$.

- Commutativity: $(v+U)+(w+U) = (v+w)+U$
$= (w+v)+U$
$= (w+U)+(v+U)$

- Associativity: $((v+U)+(w+U))+(x+U) = ((v+w)+U)+(x+U)$
$= ((v+w)+x)+U$
$= (v+(w+x))+U$
$= (v+U)+((w+x)+U)$
$= (v+U)+((w+U)+(x+U))$

- Additive identity: $(v+U)+(0+U) = (v+0)+U$
$= v+U.$

- Additive inverse: $(v+U)+((-v)+U) = (v+(-v))+U$
$= 0+U.$

- Multiplicative identity: $1(v+U) = (1v)+U$
$= v+U.$

- Distributive properties:
$$a((v+U)+(w+U)) = a((v+w)+U)$$
$$= a(v+w)+U$$
$$= (av+aw)+U$$
$$= ((av)+U)+((aw)+U)$$
$$= a(v+U)+a(w+U).$$

and
$$(a+b)(v+U) = ((a+b)v)+U$$
$$= (av+bv)+U$$
$$= ((av)+U)+((bv)+U)$$
$$= a(v+U)+b(v+U).$$
∎

Defn 3.88 Let U be a subspace of V. The quotient map is the linear map $\pi: V \to V/U$ defined by
$$\pi(v) = v+U \quad \text{for all } v \in V.$$

~~Thm~~ 3.89 Dimension of a quotient space

Suppose V is finite-dimensional and U is a subspace of V. Then
$$\dim V/U = \dim V - \dim U.$$

proof: Let $\pi : V \to V/U$ be the quotient map.

First, we claim $\operatorname{null} \pi = U$.

since $v \in U$, we have $v - 0 = v \in U$, so by 3.85 of Axler,
$$v + U = 0 + U$$

In fact, we have
$$\pi(v) = v + U$$
$$= 0 + U.$$

So $v \in \operatorname{null}\pi$, and so $U \subset \operatorname{null}\pi$.

If $v \in \operatorname{null}\pi$, then $\pi(v) = 0 + U$.

since we also have $\pi(v) = v + U$,

we conclud $v + U = 0 + U$.

By 3.85 of Axler,
$$v = v - 0 \in U.$$

So $\operatorname{null}\pi \subset U$.

Therefore, we conclude the set equality
$$\operatorname{null}\pi = U.$$

Next claim: $\operatorname{range}\pi = V/U$.

Let $w \in \operatorname{range}\pi$. Then $w = \pi(v)$ for some $v \in U$

In fact, by definition 3.88, we have
$$w = \pi(v)$$
$$= v + U$$
$$\in V/U$$

So we get $\operatorname{range}\pi \subset V/U$.

Suppose we have $v + U \in V/U$. By definition 3.88,
$$v + U = \pi(v)$$
$$\in \operatorname{range}\pi$$

So $V/U \subset \operatorname{range}\pi$.

Therefore, $\operatorname{range}\pi = V/U$.

By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have
$$\dim V = \dim \operatorname{null}\pi + \dim \operatorname{range}\pi$$
$$= \dim U + \dim V/U$$
as desired. $\boxed{\equiv}$

Defn 3.90 Suppose $T \in \mathcal{L}(V,W)$. Define $\tilde{T} : V/(\operatorname{null} T) \to W$ by
$$\tilde{T}(v + \operatorname{null} T) = Tv.$$

Show that $\tilde{T}$ makes sense ($\tilde{T}$ is well-defined)

Suppose $u, v \in V$ satisfy
$$u + \operatorname{null}T = v + \operatorname{null}T$$

By 3.85 of Axler, we have
$$u - v \in \operatorname{null}T$$

This means $T(u-v) = 0$

In fact, we have
$$Tu - Tv = T(u-v) = 0,$$

so $Tu = Tv$.

Therefore, we conclude
$$\tilde{T}(u + \operatorname{null}T) = Tu$$
$$= Tv$$
$$= \tilde{T}(v + \operatorname{null}T)$$

and so $\tilde{T}$ is well-defined.

Defn 3.91 ~~Null space and range of $\tilde{T}$~~

Suppose $T \in \mathcal{L}(V,W)$. Then

(a) $\tilde{T} : V/(\operatorname{null}T) \to W$ is a linear map; $\tilde{T} \in \mathcal{L}(V/(\operatorname{null}T), W)$

(b) $\tilde{T}$ is injective

(c) $\operatorname{range}\tilde{T} = \operatorname{range}T$

(d) $V/(\operatorname{null}T)$ is isomorphic to $\operatorname{range}T$.

proof: (a) Let $u, v \in V$ and $\lambda \in \mathbb{F}$.

- additivity: $\tilde{T}((u + \operatorname{null}T) + (v + \operatorname{null}T))$
$$= \tilde{T}((u+v) + \operatorname{null}T)$$

→

$= T(u+v)$
$= Tu + Tv$
$= \tilde{T}(u+\text{null}T) + \tilde{T}(v+\text{null}T)$

• Homogeneity: $\tilde{T}(\lambda(v+\text{null}T)) = \tilde{T}((\lambda v)+\text{null}T)$
$= T(\lambda v)$
$= \lambda Tv$
$= \lambda\tilde{T}(v+\text{null}T)$.

Therefore, $\tilde{T}$ is linear.

(b) Suppose $v \in V$ satisfies $\tilde{T}(v+\text{null}T) = 0$
then we have
$Tv = \tilde{T}(v+\text{null}T)$
$= 0$
So $v - 0 = v \in \text{null}T$
By 3.85 of Axler,
$v + \text{null}T = 0 + \text{null}T$ additive identity of $V/(\text{null}T)$

~~By part (a), $\tilde{T}$ is linear. So 3.11 of Axler, we have $\tilde{T}(0+\text{null}T) = 0$.~~

~~So $0+\text{null}T \in \text{null}\tilde{T}$, or $\{0+\text{null}T\} \subset \text{null}\tilde{T}$~~.

Therefore, $\text{null}\tilde{T} \subset \{0+\text{null}T\}$
But $\tilde{T}(0+\text{null}T) = 0$ since $\tilde{T}$ is linear.
So $\{0+\text{null}T\} \subset \text{null}\tilde{T}$
So $\text{null}\tilde{T} = \{0+\text{null}T\}$
By 3.16 of Axler, $\tilde{T}$ is injective

(c) For all $v \in V$, $\tilde{T}(v+\text{null}T) = Tv$.
Suppose $w \in \text{range}T$. Then $w = Tw$ for some $v \in V$.
In fact,
$w = Tv$
$= \tilde{T}(v+\text{null}T)$
$\in \text{range}\tilde{T}$.
So $\text{range}T \subset \text{range}\tilde{T}$
Suppose $x \in \text{range}\tilde{T}$. Then $x = \tilde{T}(v+\text{null}T)$ for some $v \in V$
In fact,
$x = \tilde{T}(v+\text{null}T)$
$= Tv$
$\in \text{range}T$.
So $\text{range}\tilde{T} \subset \text{range}T$.
Therefore, we conclude $\text{range}\tilde{T} = \text{range}T$.

(d) By part (c), $\text{range}\tilde{T} = \text{range}T$.
If we think of $\tilde{T}$ as a map into $\text{range}\tilde{T}$,
$\tilde{T}: V/(\text{null}T) \to \text{range}T$ is surjective
So $\tilde{T}$ is also surjective.
By part (b), $\tilde{T}$ is also injective.
Therefore, by 3.56 of Axler, $\tilde{T}$ is invertible.
By part (a), $\tilde{T}$ is linear.
Therefore, $\tilde{T}: V/(\text{null}T) \to \text{range}T$ is an isomorphism.

$\blacksquare$