3.E Products & Quotients of vector spaces

7/17/19 Wed. week4

3.71 Def

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$

- The product $V_1 \times \dots \times V_m$ is defined by $V_1 \times \dots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}$
- Addition on $V_1 \times \dots \times V_m$ is defined by $(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m)$
- Scalar multi. on $V_1 \times \dots \times V_m$ is defined by $\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m)$

3.72 Example: $\underbrace{(5 - 6x + 4x^2, (3, 8, 7))}_{\text{length } 2} \in P_2(\mathbb{R}) \times \mathbb{R}^3$

Example: $\underbrace{((1, 2), (3, 4, 5))}_{\text{length } 2} \in \mathbb{R}^2 \times \mathbb{R}^3$

Example: $\underbrace{(1, (2, 3), (4, 5))}_{\text{length } 3} \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$

3.73 Product of vector spaces is a vector space

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$. Then $V_1 \times \dots \times V_m$ is a vector space over $\mathbb{F}$.

Proof: Let $u_i, v_i, w_i \in V_i$ for each $i = 1, \dots, m$, & let $\lambda \in \mathbb{F}$

- commutativity: Since V_i is a vector space, we have $u_i + v_i = v_i + u_i$ so we have $(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m) = (v_1 + u_1, \dots, v_m + u_m) = (v_1, \dots, v_m) + (u_1, \dots, u_m)$
- associativity: Since V_i is a vector space, we have $(u_i + v_i) + w_i = u_i + (v_i + w_i)$

  so we have

  $((u_1, \dots, u_m) + (v_1, \dots, v_m)) + (w_1, \dots, w_m) = (u_1 + v_1, \dots, u_m + v_m) + (w_1, \dots, w_m)$

  $= ((u_1 + v_1) + w_1, \dots, (u_m + v_m) + w_m)$

  $= (u_1 + (v_1 + w_1), \dots, u_m + (v_m + w_m))$

  $= (u_1, \dots, u_m) + (v_1 + w_1, \dots, v_m + w_m)$

  $= (u_1, \dots, u_m) + ((v_1, \dots, v_m) + (w_1, \dots, w_m))$
- Additive Identity: we have $(0, \dots, 0) \in V_1 \times \dots \times V_m$. And it satisfies $(v_1, \dots, v_m) + (0, \dots, 0) = (v_1 + 0, \dots, v_m + 0)$

  $= (v_1, \dots, v_m)$

  so $(0, \dots, 0)$ is the additive identity of $V_1 \times \dots \times V_m$

- **Additive Inverse**: we have $(-v_1, \ldots, -v_m) \in V_1 \times \cdots \times V_m$ & it satisfies
$$(v_1, \ldots, v_m) + (-v_1, \ldots, -v_m) = (v_1 + (-v_1), \ldots, v_m + (-v_m))$$
$$= (v_1 - v_1, \ldots, v_m - v_m)$$
$$= (0, \ldots, 0)$$
So $(-v_1, \ldots, -v_m)$ is the additive inverse of $V_1 \times \cdots \times V_m$.
- **Multiplicative Identity**: we have
$$1(v_1, \ldots, v_m) = (1v_1, \ldots, 1v_m)$$
$$= (v_1, \ldots, v_m)$$

- **Distributive prop** for all $a, b \in \mathbb{F}$, we have
$$a((u_1, \ldots, u_m) + (v_1, \ldots, v_m)) = a(u_1 + v_1, \ldots, u_m + v_m)$$
$$= (a(u_1 + v_1), \ldots, a(u_m + v_m))$$
$$= (au_1 + av_1, \ldots, au_m + av_m)$$
$$= (av_1, \ldots, au_m) + (av_1, \ldots, av_m)$$
$$= a(u_1, \ldots, u_m) + a(v_1, \ldots, v_m)$$

and $(a+b)(v_1, \ldots, v_m) = ((a+b)v_1, \ldots, (a+b)v_m)$
$$= (av_1 + bv_1, \ldots, av_m + bv_m)$$
$$= (av_1, \ldots, av_m) + (bv_1, \ldots, bv_m)$$
$$= a(v_1, \ldots, v_m) + b(v_1, \ldots, v_m)$$

3.79 **Example** show that $\mathbb{R}^2 \times \mathbb{R}^3$ is isomorphic to $\mathbb{R}^5$

Note that, as vector spaces, $\mathbb{R}^2 \times \mathbb{R}^3 \neq \mathbb{R}^5$ b/c elements $\underbrace{((x_1, x_2), (x_3, x_4, x_5))}_{\text{length } 2}$ of $\mathbb{R}^2 \times \mathbb{R}^3$ have length 2 but elements $\underbrace{(x_1, x_2, x_3, x_4, x_5)}_{\text{length } 5}$ of $\mathbb{R}^5$ have length 5.

Proof: Define $T: \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$ by $T((x_1, x_2), (x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4$

First, we will show that T is injective. Let $((x_1, x_2), (x_3, x_4, x_5)) \in \text{null}\, T$, which means $T((x_1, x_2), (x_3, x_4, x_5)) = (0, 0, 0, 0, 0)$.

Then we have $(0, 0, 0, 0, 0) = T((x_1, x_2), (x_3, x_4, x_5))$
$$= (x_1, x_2, x_3, x_4, x_5)$$
So $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0$.

This means $((x_1, x_2), (x_3, x_4, x_5)) = ((0, 0), (0, 0, 0))$

So $\text{null}\, T \subset \{(0, 0), (0, 0, 0)\}$

$T((0,0), (0,0,0)) = (0,0,0,0,0)$

we also have $\{((0,0), (0,0,0))\} \subset \text{null}\, T$

Therefore, $\text{null}\, T = \{((0,0), (0,0,0))\}$

By 3.16 of Axler, T is injective

Next, we will show that T is surjective
For all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, we have
$(x_1, x_2, x_3, x_4, x_5) = T((x_1, x_2), (x_3, x_4, x_5)) \in$ range T
so $\mathbb{R}^5 \subset$ range T. But range T is a subspace of $\mathbb{R}^5$. So
we have range $T = \mathbb{R}^5$. So T is surjective
Therefore, by 3.56 of Axler, T is invertible
Next, we will show that T is linear
- Additivity: For all $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}$, we have
$$T(((x_1, x_2), (x_3, x_4, x_5)) + ((y_1, y_2), (y_3, y_4, y_5)))$$
$$= T((x_1+y_1, x_2+y_2), (x_3+y_3, x_4+y_4, x_5+y_5))$$
$$= (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5)$$
$$= (x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)$$
$$= T(((x_1, x_2), (x_3, x_4, x_5))) + T(((y_1, y_2), (y_3, y_4, y_5)))$$

- Homogeneity: for all $\lambda \in F$ & for all $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$, we have
$$T(\lambda((x_1, x_2), (x_3, x_4, x_5))) = T(((\lambda x_1, \lambda x_2), (\lambda x_3, \lambda x_4, \lambda x_5)$$
$$= (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5)$$
$$= \lambda(x_1, x_2, x_3, x_4, x_5)$$
$$= \lambda T(((x_1, x_2), (x_3, x_4, x_5)))$$

Therefore T is linear
So T is invertible & linear. Therefore, T is an isomorphism

3.75 example
Find a basis of $P_2(\mathbb{R}) \times \mathbb{R}^2$
soln: $(1, (0,0)), (x, (0,0)), (x^2, (0,0)), (0, (1,0)), (0, (0,1))$
$1, x, x^2$ is a basis of $P_2(\mathbb{R})$ $\quad (1,0), (0,1)$ is a basis of $\mathbb{R}^2$

3.76 Dimension of a product is the sum of dimensions
Suppose $V_1, \ldots, V_m$ are finite-dimensional vector spaces.
Then $V_1 \times \cdots \times V_m$ is finite-dim & $\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V$
Proof: Axler
Choose a basis of each V_j. For each basis vector of each V_j, consider the element of $V_1 \times \cdots \times V_m$ that equals the basis vector in the jth slot & 0 in other slots. The list of all such vectors is lin indep. & spans $V_1 \times \cdots \times V_m$. Therefore, it's a basis of $V_1 \times \cdots \times V_m$, w/ length $\dim V_1 + \cdots + \dim V_m$
Ryan's Interpretation:
Let $j = 1, \ldots, m$. Let $v_{j,1}, \ldots, v_{j,n}$ be a basis of each V_j. Then $n_j = \dim V_j$, & the ith basis vector of V_j is $v_{j,i}$ for $i = 1, \ldots, n_j$. So we have ?
$(v_{1,1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, v_{1,n_1})$, length: $n_1 = \dim V_1$
$(v_{2,1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, v_{2,n_2})$, length: $n_2 = \dim V_2$
⋮

$(v_{m,1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, v_{m, n_m})$, length: $n_m = \dim V_m$
is a basis of $V_1 \times \cdots \times V_m$

Total length
$n_1 + n_2 + \cdots + n_m$
$= \dim V_1 + \dim V_2 + \cdots + \dim V_m$

Products & Direct sums

3.77 Products & Direct Sums

Suppose that $U_1, \ldots, U_m$ are subspaces of V.
Define a linear map
$\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$ by $\Gamma(u_1, \ldots, u_m) = u_1 + \cdots + u_m$
Then $U_1 + \cdots + U_m$ is a direct sum if and only if Γ is injective

Proof:

Forward direction: If $U_1 + \cdots + U_m$ is a direct sum, then Γ is injective. Suppose $(u_1, \ldots, u_m) \in \text{null}\,\Gamma$, so that
$\Gamma(u_1, \ldots, u_m) = \underbrace{0 + \cdots + 0}_{m}$
Since $U_1 + \cdots + U_m$ is a direct-sum, by 1.44 of Axler, the only way to write the zero vector $0 + \cdots + 0$ is to take $u_1 = 0, \ldots, u_m = 0$. So $(u_1, \ldots, u_m) = 0$'s so $\text{null}\,\Gamma = \{0\}$.
By 3.16 of Axler Γ is injective.

Backward direction: If Γ is injective, then $U_1 + \cdots + U_m$ is a direct sum. Since Γ is injective, by 3.16 of Axler we have $\text{null}\,\Gamma = \{(\underset{u_1}{0}, \ldots, \underset{u_m}{0})\}$ so the only way to write $0 + \cdots + 0$ is to take $u_1 = 0, \ldots, u_m = 0$. By 1.44 of Axler, $U_1 + \cdots + U_m$ is a direct sum

3.78 A sum is a direct sum if & only if dimensions add up

Suppose V is finite-dim & $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if & only if
$$\dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m$$

Proof: By the proof of 3.77 of Axler, the map $\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$ defined by $\Gamma(u_1, \ldots, u_m) = u_1 + \cdots + u_m$ is surjective. So the fundamental thm of linear maps (3.22 of Axler) gives us

$\dim(U_1 + \cdots + U_m) = \dim \text{range}\,\Gamma$ (b/c Γ is surjective, $\text{range}\,\Gamma = U_1 + \cdots + U_m$)

$= \dim(U_1 \times \cdots \times U_m) - \dim \text{null}\,\Gamma$ by fun. Thm of lin maps

$= \dim(U_1 \times \cdots \times U_m) - \dim\{0\}$ if & only if Γ is injective (3.16 of Axler)

$= \dim(U_1 \times \cdots U_m)$

if & only if Γ is injective.

combine w/ 3.77 & 3.76 of Axler to conclude that $U_1 + \cdots + U_m$ is a direct sum if & only if we have

$$\dim(U_1 + \cdots + U_m) = \dim(U_1 \times \cdots \times U_m)$$
$$= \dim U_1 + \cdots + \dim U_m \text{ by 3.76 of Axler}$$

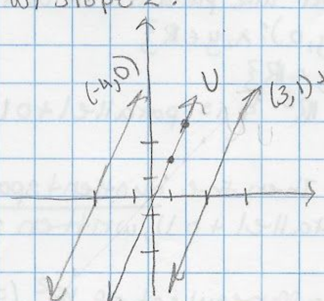
<u>Quotient of Vector spaces</u>

<u>3.79 Def</u>

Suppose $v \in V$ & U is a subspace of V. Then $v+U$ is the subset of V defined $v + U = \{v + u : u \in U\}$.

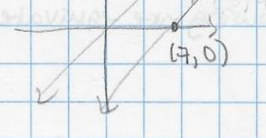
<u>3.80 Example</u>

Let $V = \mathbb{R}^2$ & $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Then U is the line in $\mathbb{R}^2$ through the origin w/ slope 2.



So $(3,1) + U$ is a line in $\mathbb{R}^2$ that contains the point $(3,1)$ & has slope & $(-4,0) + U$ is a line in $\mathbb{R}^2$ that contains the point $(-4,0)$ & has slope 2.

$$(3,1) + U = \{(3,1) + (x, 2x) : x \in \mathbb{R}\}$$
$$= \{(3+x, 1+2x) : x \in \mathbb{R}\}$$
$$(-4,0) + U = \{(-4,0) + (x, 2x) : x \in \mathbb{R}\}$$
$$= \{(-4+x, 2x) : x \in \mathbb{R}\}$$



Prove: Since $(7,0)$ & $(17,20)$ lie on the same line, $(7,0) + U = (17,20) + U$

Proof: $(17,20) + U = \{(17,20) + (x, 2x) : (x, 2x) \in U\}$
$$= \{(17 + x, 20 + 2x) : x \in \mathbb{R}\}$$

$$(7,0) + U = \{(7,0) + (x, 2x) : (x, 2x) \in U\}$$
$$= \{(7+x, 2x) : x \in \mathbb{R}\}$$

let $y = x - 10$ $\quad = \{(17 - 10 + x, 20 - 20 + 2x) : x \in \mathbb{R}\}$

since $x \in \mathbb{R}$, $\quad = \{(17 - (x-10), 20 - 2(x-10)) : x \in \mathbb{R}\}$

it follows $\quad = \{(17 + y, 20 + 2y) : y \in \mathbb{R}\}$

that $y \in \mathbb{R}$ $\quad = \{(17, 20) + (y, 2y) : y \in \mathbb{R}\}$

$$= (17, 20) + U$$

3.81 Def

- An affine subset of V is a subset of V of the form $v+U$ for some $v \in V$ & some subspace U of V.
- If U is a subset of V, for all $v \in V$, the affine subset $v+U$ is said to be parallel to U.

3.82 Example

- Let $V = \mathbb{R}^2$ & $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, as in example 3.80. Then all the lines in $\mathbb{R}^2$ w/ slope 2 are parallel to U. & these lines are affine subsets in $\mathbb{R}^2$
- Let $V = \mathbb{R}^3$ & $U = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$. Then the affine subsets of $\mathbb{R}^3$ are all the planes in $\mathbb{R}^3$ that are parallel to U. For example,

$$(0,0,2)+U = \{(0,0,2)+(x,y,0) : x, y \in \mathbb{R}\}$$
$$= \{(x,y,2) : x, y \in \mathbb{R}\}$$

is an affine subset of $\mathbb{R}^3$ & is parallel to U.

3.83 def.

Let U be a subspace of V. Then the quotient space V/U is the set of all affine subsets of V parallel to U, written:

quotient space → $V/V = \{v+U : v \in V\}$

Example: $(7,0)+U$ is an affine subset of $\mathbb{R}^2$ $(7,0)+U \in \mathbb{R}^2/U$

3.84 Example

- If $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, then $\mathbb{R}^2/U$ is the set of all lines in $\mathbb{R}^2$ that has slope 2
- If U is a line in $\mathbb{R}^3$ containing the origin, then $\mathbb{R}^3/U$ is the set of all lines in $\mathbb{R}^3$ parallel to U. For example, $U = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$

$$\mathbb{R}^3/U_1 = \{(0,0,z)+U_1 : x, y, z \in \mathbb{R}\}$$
$$U_2 = \{(0,y,z) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$$
$$\mathbb{R}^3/U_2 = \{(x,0,0)+U_2 : x, y, z \in \mathbb{R}\}$$

IMPORTANT RESULT FOR UPCOMING EXAMS!!

3.85 Two affine subsets parallel to U are equal or disjoint

Let V be a subspace of V & $v, w \in V$. Then the following are equivalent:

a) $v - w \in U$

b) $v+U = w+U$

c) $(v+U) \cap (w+U) \neq \emptyset$

Proof:

a) implies b):

Suppose a) holds: $v-w \in U$. Let $u \in U$ be arbitrary. Since U is a subspace of V, in particular it is closed under addition. Since $u \in U$ & $v-w \in U$, we have $(v-w)+u \in U$. For all $u \in U$, we have $v+u = w+v-w+u \in U = w+((v-w)+u) \in w+U$,

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Similarly, for all $u \in U$, we have $\quad$ Therefore, $v + U \subset w + U$.

$$\begin{aligned} w + u &= v + w - v + u \\ &= v + (-(v-w) + u) \quad \in U \\ &\in v + U \end{aligned}$$

Therefore, $w + U \subset v + U$.

So we conclude the set equality $v + U = w + U$, which is b)

b) implies c) :

Suppose b) holds : $v + U = w + U$. There exists $u \in U$ that satisfies

$$\begin{aligned} v + u &\in v + U \\ &= w + U \end{aligned}$$

So $v + u \in v + U$ and $v + u \in w + U$

That is, $v + u \in (v + U) \cap (w + U)$

In other words, $(v + u) \cap (w + U) \neq \emptyset$, which is c)

c) implies a) :

Suppose c) holds : $(v + U) \cap (w + U) \neq \emptyset$. Then there exist $u_1, u_2 \in U$ that satisfies

$$v + u_1 = w + u_2$$

Since U is a subspace of V, it is closed under addition & scalar mult., which means $u_1 - u_2 \in U$. In fact we have,

$$\begin{aligned} v - w &= u_2 - u_1 \\ &= -(u_1 - u_2) \\ &\in U, \text{ which is a)} \end{aligned}$$

3.86 Def

Let U be a subspace of V. Then:

- addition is defined on V/U by $(v + U) + (w + U) = (v + w) + U$
- scalar mult. is defined on V/U by $\lambda(v + U) = (\lambda v) + U$

3.87 Quotient space is a vector space

Let U be a subspace of V. Then V/U is a vector space w/respect to the operations defined in def 3.86

Proof: Let $v, w \in V$ be arbitrary

First, we need to show that the operations of addition & scalar mult. make sense on V/U.

Suppose $\hat{v}, \hat{w} \in V$ satisfy $v + U = \hat{v} + U$ & $w + U = \hat{w} + U$

First, we will show that addition makes sense on V/U. Since U is a subspace of V, it is closed under addition. So

$$(v + w) - (\hat{v} + \hat{w}) = v - \hat{v} + w - \hat{w} \in U$$

By 3.85 of Axler, $(v + w) + U = (\hat{v} + \hat{w}) + U$

So addition makes sense on V/U

Now let $\lambda \in \mathbb{F}$ be arbitrary. Suppose $\hat{v} \in V$ satisfies $v + U = \hat{v} + U$

By 3.85 of Axler, $v - \hat{v} \in U$. Since U is a subspace of V, it is closed under scalar mult., which means $\lambda(u - \hat{u}) \in U$.

so we have,

$$\lambda v - \lambda \hat{v} = \lambda(v - \hat{v}) \in U.$$

By 3.85 of Axler

$$\lambda v + U = \lambda \hat{v} + U$$

so scalar mult. makes sense on V/U

Next we will show that V/U satisfies all axioms of a vector space.

Let $v, w, x \in V$ & $\lambda, a, b \in \mathbb{F}$. be arbitrary

- Commutativity: $(v+U)+(w+U) = (v+w)+U$

$$= (w+v)+U$$
$$= (w+U)+(v+U)$$

- Associativity: $((v+U)+(w+U))+(x+U) = ((v+w)+U)+(x+U)$

$$= ((v+w)+x)+U$$
$$= (v+(w+x))+U$$
$$= (v+U)+((w+x)+U)$$
$$= (v+U)+((w+U)+(x+U))$$

- Additive Identity:

$$(v+u)+(0+U) = (v+0)+U$$
$$= v+U$$

- Additive Inverse: $(v+U)+((-v)+U) = (v+(-v))+U$

$$= 0+U$$

- Multiplicative Identity: $1(v+U) = (1v)+U$

$$= v+U$$

- Distributive Prop: $a((v+U)+(w+U)) = a((v+w)+U)$

$$= a(v+w)+U$$
$$= (av+aw)+U$$
$$= ((av)+U)+((aw)+U)$$
$$= a(v+U)+a(w+U)$$

and $(a+b)(v+u) = ((a+b)v)+U$

$$= (av+bv)+U$$
$$= ((av)+U)+((bv)+U)$$
$$= a(v+U)+b(v+U)$$

<u>3.88 Def</u>

Let U be a supspace of V. The <u>quotient map</u> is the linear map $\pi : V \to V/U$ defined by

$$\pi(v) = v+U \text{ for all } v \in V$$

<u>3.89 Dimension of a quotient space</u>

Suppose V is finite-dimensional & U is a subspace of V.

Then $\dim V/U = \dim V - \dim U$

<u>Proof:</u> Let $\pi : V \to V/U$ be the quotient map

First we claim null $\pi = U$.

Since $v \in U$, we have $v - 0 = v \in U$, so by 3.85 of Axler,

$$v + U = 0 + U$$

In fact, we have

$$\pi(v) = v + U$$
$$= 0 + U$$

so $v \in \text{null } \pi$, & so $U \subset \text{null } \pi$.

If $v \in \text{null } \pi$, then $\pi(v) = 0 + U$

Since we also have $\pi(v) = v + U$,

we conclude $v + U = 0 + U$

By 3.85 of Axler,

$$v = v - 0 \in U$$

so $\text{null } \pi \subset U$.

Therefore, we conclude the set equality

$$\text{null } \pi = U$$

Next claim: range $\pi = V/U$.

Let $w \in \text{range } \pi$. Then $w = \pi(v)$ for some $v \in V$. In fact, by def. 3.88, we have $w = \pi(v) = v + U \in V/U$. So we get range $\pi \subset V/U$.

Suppose we have $v + U \in V/U$. By def. 3.88, $v + U = \pi(v) \in \text{range } \pi$. So $V/U \subset$ range π.

Therefore, range $\pi = V/U$.

By the fundamental Thm of linear maps (3.22 of Axler), we have

$$\dim V = \dim \text{null } \pi + \dim \text{range } \pi$$
$$= \dim U + \dim V/U \text{ as desired}$$

7/23/19 Tues week 5 **3.90 Def.**

Suppose $T \in \mathcal{L}(V,W)$. Define $\tilde{T} : V/(\text{null } T) \to W$ by $\tilde{T}(v + \text{null } T) = Tv$.

end of def.

Show that $\tilde{T}$ makes sense ($\tilde{T}$ is well-defined). Suppose $u, v \in V$ satisfy $u + \text{null } T = v + \text{null } T$

By 3.85 of Axler, we have $u - v \in \text{null } T$,

This means $T(u - v) = \emptyset$.

In fact, we have $Tu - Tv = T(u - v) = \emptyset$, so $Tu = Tv$

Therefore, $\tilde{T}(u + \text{null } T) = Tu$

$$= Tv$$
$$= \tilde{T}(v + \text{null } T)$$, & so $\tilde{T}$ is well-defined

3.91 Null space & range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V,W)$. Then:

a) $\tilde{T}$ $V/(\text{null } T) \to W$ is a linear map; $\tilde{T} \in \mathcal{L}(V/(\text{null } T), W)$

b) $\tilde{T}$ is injective

c) range $\tilde{T}$ = range T

d) $V/(\text{null } T)$ is isomorphic to range T

Proof:

a) Let $u, v \in V$ & $\lambda \in \mathbb{F}$

- Additivity: $\tilde{T}((u+\text{null}\,T)+(v+\text{null}\,T))$
$$= \tilde{T}((u+v)+\text{null}\,T)$$
$$= T(u+v)$$
$$= Tu + Tv$$
$$= \tilde{T}(u+\text{null}\,T) + \tilde{T}(v+\text{null}\,T)$$

- Homogeneity: $\tilde{T}(\lambda(v+\text{null}\,T)) = \tilde{T}(\lambda v + \text{null}\,T)$
$$= T(\lambda v)$$
$$= \lambda T v$$
$$= \lambda \tilde{T}(v+\text{null}\,T)$$

Therefore, $\tilde{T}$ is linear.

b) Suppose $v \in V$ satisfies $\tilde{T}(v+\text{null}\,T) = \emptyset$.
and
$$Tv = \tilde{T}(v+\text{null}\,T)$$
$$= \emptyset$$
So $v - \emptyset = v \in \text{null}\,T$

By 3.85 of Axler, $v+\text{null}\,T = \emptyset + \text{null}\,T$ ← additive identity of $V/(\text{null}\,T)$

therefore, $\text{null}\,\tilde{T} \subset \{0+\text{null}\,T\}$. But $\tilde{T}(0+\text{null}\,T) = \emptyset$ since $\tilde{T}$ is linear.

So $\{0+\text{null}\,T\} \subset \text{null}\,\tilde{T}$. So $\text{null}\,\tilde{T} = \{0+\text{null}\,T\}$

→ By 3.16 of Axler, $\tilde{T}$ is injective

c) For all $v \in V$, $\tilde{T}(v+\text{null}\,T) = Tv$

Suppose $w \in \text{range}\,T$. Then $w = Tv$ for some $v \in V$. In fact, $w = Tv$
$$= \tilde{T}(v+\text{null}\,T) \in \text{range}\,\tilde{T}$$
So $\text{range}\,T \subset \text{range}\,\tilde{T}$

Suppose $x \in \text{range}\,\tilde{T}$. Then $x = \tilde{T}(v+\text{null}\,T)$ for some $v \in V$.

In fact, $x = \tilde{T}(v+\text{null}\,T)$
$$= Tv$$
$$\in \text{range}\,T$$
So $\text{range}\,\tilde{T} \subset \text{range}\,T$

Therefore, we conclude $\text{range}\,\tilde{T} = \text{range}\,T$

d) By part c), $\text{range}\,\tilde{T} = \text{range}\,T$

If we think of $\tilde{T}$ as a map into $\text{range}\,\tilde{T}$,
$\tilde{T}: V/(\text{null}\,T) \to \text{range}\,T$ is surjective.

So $\tilde{T}$ is also surjective.

By part b), $\tilde{T}$ is also injective, Therefore, by 3.56 of Axler, $\tilde{T}$ is invertible. By part a), $\tilde{T}$ is linear. Therefore, $\tilde{T}: V/(\text{null}\,T) \to \text{range}\,T$ is an isomorphism

3.F Duality

3.92 def

A linear functional on V is a linear map $\varphi: V \to \mathbb{F}$. In other words, $\varphi \in \mathcal{L}(V, \mathbb{F})$

3.93 Example

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

- Define $\varphi: \mathbb{R}^3 \to \mathbb{R}$ by
  $$\varphi(x,y,z) = 4x - 5y + 2z$$
  Then φ is a linear functional on $\mathbb{R}^3$
- For some $c_1, \ldots, c_n \in \mathbb{F}$, the map $\varphi: \mathbb{F}^n \to \mathbb{F}$ defined by
  $$\varphi(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$$
  then φ is a linear functional on $\mathbb{F}^n$
- Define $\varphi: P(\mathbb{R}) \to \mathbb{R}$ by
  $$\varphi(p) = \int_0^1 p(x)\,dx$$
  Then φ is a linear functional on $P(\mathbb{R})$

3.94 Def.

The dualspace of V, denoted V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbb{F})$

3.95 $\dim V' = \dim V$

Suppose V is finite-dim. Then V' is also finite-dim & $\dim V' = \dim V$

Proof:
$$\begin{aligned}
\dim V' &= \dim \mathcal{L}(V, \mathbb{F}) \\
&= (\dim V)(\dim \mathbb{F}) \quad \text{by 3.61 of Axler} \\
&= (\dim V) \cdot 1 \\
&= \dim V
\end{aligned}$$

3.96 Def

If $v_1, \ldots, v_n$ is a basis of V, then the dual basis of $v_1, \ldots, v_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j for any $j = 1, \ldots, n$ is a linear functional on V that satisfies
$$\varphi_j(v_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

3.97 Example

what is the dual basis of the standard basis $e_1, \ldots, e_n$ of $\mathbb{F}^n$?
$$e_1 = (1, 0, \ldots, 0)$$
$$e_2 = (0, 1, 0, \ldots, 0)$$