07-17-19

3E Products and Quotients of Vector Spaces

3.71 Definition

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$

- The product $V_1 \times \cdots \times V_m$ is defined by $V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}$
- Addition on $V_1 \times \cdots \times V_m$ is defined by $(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1 + \cdots + u_m + v_m)$
- Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by $\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m)$

3.72 Example

- $\underbrace{(5 - 6x + 4x^2, (3, 8, 7))}_{\text{length } 2} \in P_2(\mathbb{R}) \times \mathbb{R}^3$
- $\underbrace{((1, 2), (3, 4, 5))}_{\text{length } 2} \in \mathbb{R}^2 \times \mathbb{R}^3$
- $\underbrace{(1, (2, 3), (4, 5))}_{\text{length } 3} \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$

3.73 Product of vector spaces is a vector space

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$

Proof: Let $u_i, v_i, w_i \in V_i$ for each $i = 1, \dots, m$, and let $\lambda \in \mathbb{F}$

- community: Since each V_i is a vector space, we have $u_i + v_i = v_i + u_i$

  So we have

$$
\begin{aligned}
(u_1, \dots, u_m) + (v_1, \dots, v_m) &= (u_1 + v_1, \dots, u_m + v_m) \\
&= (v_1 + u_1, \dots, v_m + u_m) \\
&= (v_1, \dots, v_m) + (u_1, \dots, u_m)
\end{aligned}
$$

- associativity: Since V_i is a vector space, we have

$$(u_i + v_i) + w_i = u_i + (v_i + w_i)$$

  So we have

$$
\begin{aligned}
((u_1, \dots, u_m) + (v_1, \dots, v_m)) + (w_1, \dots, w_m) &= (u_1 + v_1, \dots, u_m + v_m) + (w_1, \dots, w_m) \\
&= ((u_1 + v_1) + w_1, \dots, (u_m + v_m) + w_m) \\
&= (u_1 + (v_1 + w_1), \dots, u_m + (v_m + w_m)) \\
&= (u_1, \dots, u_m) + (v_1 + w_1, \dots, v_m + w_m) \\
&= (u_1, \dots, v_m) + ((v_1, \dots, v_m) + (w_1, \dots, w_m))
\end{aligned}
$$

- Additive identity: we have $\underset{\text{zero vectors}}{(0, \dots, 0)} \in V_1 \times \cdots \times V_m$. And it satisfies

$$
\begin{aligned}
(v_1, \dots, v_n) + (0, \dots, 0) &= (v_1 + 0, \dots, v_m + 0) \\
&= (v_1, \dots, v_m)
\end{aligned}
$$

  So $(0, \dots, 0)$ is the additive identity of $V_1 \times \cdots \times V_m$

- Additive inverse: we have $(-v_1, \dots, v_n) \in V_1 \times \cdots \times V_m$. And it satisfies

$$(v_1, \dots, v_m) + (-v_1, \dots, -v_m) = (v_1 + (-v_1), \dots, v_m + (-v_m))$$
$$= (v_1 - v_1, \dots, v_m - v_m)$$
$$= (0, \dots, 0)$$

So $(v_1, \dots, -v_m)$ is the additive inverse of $V_1 \times \cdots \times V_m$

- multiplicative identity: we have

$$1(v_1, \dots, v_m) = (1v_1, \dots, 1v_m)$$
$$= (v_1, \dots, v_m)$$

- Distributive properties: For all $a, b \in \mathbb{F}$ we have

$$a((u_1, \dots, u_m) + (v_1, \dots, v_m)) = a(u_1 + v_1, \dots, u_m + v_m)$$
$$= (a(u_1 + v_1), \dots, a(u_m + v_m))$$
$$= (au_1 + av_1, \dots, au_m + av_m)$$
$$= (au_1, \dots, au_m) + (av_1 + av_m)$$
$$= a(u_1, \dots, u_m) + a(v_1, \dots, v_m)$$

and

$$(a+b)(v_1, \dots, v_m) = ((a+b)v_1, \dots, (a+b)v_m)$$
$$= (av_1 + bv_1, \dots, av_m + bv_m)$$
$$= (av_1, \dots, av_m) + (bv_1, \dots, bv_m)$$
$$= a(v_1, \dots, v_m) + b(v_1, \dots, v_m)$$

3.74 Example Show that $\mathbb{R}^2 \times \mathbb{R}^3$ is isomorphic to $\mathbb{R}^5$

Note that, as vector spaces, $\mathbb{R}^2 \times \mathbb{R}^3 \neq \mathbb{R}^5$ because elements $\underbrace{((x_1, x_2), (x_3, x_4, x_5))}_{\text{length } 2}$ of $\mathbb{R}^2 \times \mathbb{R}^3$ has length 2 but elements $\underbrace{(x_1, x_2, x_3, x_4, x_5)}_{\text{length } 5}$ of $\mathbb{R}^5$ have length 5

Proof: Define $T: \mathbb{R}^2 \times \mathbb{R}^5 \to \mathbb{R}^5$ by

$$T((x_1, x_2), (x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4, x_5)$$

First, we will show that T is injective

Let $((x_1, x_2), (x_3, x_4, x_5)) \in \text{null } T$, which means

$$T((x_1, x_2), (x_3, x_4, x_5)) = (0, 0, 0, 0, 0)$$

Then we have

$$(0, 0, 0, 0, 0) = T((x_1, x_2), (x_3, x_4, x_5))$$
$$= (x_1, x_2, x_3, x_4, x_5)$$

So

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = 0$$

This means

$$((x_1, x_2), (x_3, x_4, x_5)) = ((0,0), (0,0,0))$$

So $\operatorname{null} T \subset \{((0,0), (0,0,0))\}$

$$T((0,0), (0,0,0)) = (0,0,0,0,0)$$

We also have $\{((0,0), (0,0,0))\} \subset \operatorname{null} T$

Therefore, $\operatorname{null} T = \{((0,0), (0,0,0))\}$

By 3.16 of Axler, T is injective

Next we will show that T is surjective

For all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$

$$(x_1, x_2, x_3, x_4, x_5) = T((x_1, x_2), (x_3, x_4, x_5)) \in \operatorname{range} T$$

So $\mathbb{R}^5 \subset \operatorname{range} T$. But $\operatorname{range} T$ is a subspace of $\mathbb{R}^5$

So we have

$$\operatorname{range} T = \mathbb{R}^5$$

So T is surjective

Therefore, by 3.56 of Axler, T is invertible

Next we will show that T is linear

- Additivity: For all $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}$, we have

$$\begin{aligned}
&T\Big(\big((x_1, x_2), (x_3, x_4, x_5)\big) + \big((y_1, y_2), (y_3, y_4, y_5)\big)\Big) \\
&= T\big(((x_1+y_1, x_2+y_2), (x_3+y_3, x_4+y_4, x_5+y_5))\big) \\
&= (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5) \\
&= (x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5) \\
&= T\big(((x_1, x_2), (x_3, x_4, x_5))\big) + T\big(((y_1, y_2), (y_3, y_4, y_5))\big)
\end{aligned}$$

- Homogeneity: For all $\lambda \in \mathbb{F}$ and for all $x_1, x_2, x_3, x_4, x_5 \in \mathbb{F}$ we have

$$\begin{aligned}
T\big(\lambda((x_1, x_2), (x_3, x_4, x_5))\big) &= T\big(((\lambda x_1, \lambda x_2), (\lambda x_3, \lambda x_4, \lambda x_5))\big) \\
&= (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) \\
&= \lambda(x_1, x_2, x_3, x_4, x_5) \\
&= \lambda T\big(((x_1, x_2), (x_3, x_4, x_5))\big)
\end{aligned}$$

Therefore, T is linear

So T is invertible and linear

Therefor, T is an isomorphism $\blacksquare$

3.75 Example Find a basis of $P_2(\mathbb{R}) \times \mathbb{R}^2$

Soln: $(1, (0,0)), (x, (0,0)), (x^2, (0,0)), (0, (1,0)), (0, (0,1))$

$1, x, x^2$ is a basis of $P_2(\mathbb{R})$ $(1,0), (0,1)$ is a basis of $\mathbb{R}^2$

3.76 Dimension of a product is the sum of dimensions

Suppose $V_1, \dots, V_m$ are finite-dimensional vector spaces. The $V_1 \times \dots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$$

Proof: AXLER

Choose a basis of each V_j. For each basis vector of each V_j consider the element of $V_1 \times \dots \times V_m$ that equals the basis vector in the j^{th} slot and 0 in the other slots. The list of all such vectors is linearly independent and spans $V_1 \times \dots \times V_m$. Therefore, it is a basis of $V_1 \times \dots \times V_m$, with length

$$\dim V_1 + \dots + \dim V_m \quad \boxtimes$$

RYAN

Let $j = 1, \dots, m$. Let $v_{j,1}, \dots, v_{j,n_j}$ be a basis of each V_j. Then $n_j = \dim V_j$, and the i^{th} basis vector of V_j is v_{j,i_j}

So we have

$(v_{1,1}, 0, \dots, 0), \dots, (0, \dots, 0, v_{1,n_1}), \}$ length: $n_1 = \dim V_1$

$(v_{2,1}, 0, \dots, 0), \dots, (0, \dots, 0, v_{2,n_2}), \}$ length: $n_2 = \dim V_2$

$\vdots$

$(v_{m,1}, 0, \dots, 0), \dots, (0, \dots, 0, v_{m,n_m}). \}$ length: $n_m = \dim V_m$

is a basis of $V_1 \times \dots \times V_m$.

Total length $n_1 + n_2 + \dots + n_m$ $= \dim V_1 + \dim V_2 + \dots + \dim V_m$

Products and Direct Sums

3-77 Products and Direct Sums

Suppose that $U_1, \dots, U_m$ are subspaces of V. Define a linear map

$$T : U_1 \times \dots \times U_m \to U_1 + \dots + U_m \text{ by}$$

$$T(u_1, \dots, u_m) = u_1 + \dots + u_m$$

Then $U_1 + \dots + U_m$ is a direct sum if and only if T is injective.

Proof: Forward direction: If $U_1 + \dots + U_m$ is a direct sum, then T is injective

Suppose $(u_1, \dots, u_m) \in \text{null}\, T$, so that

$$T(u_1, \dots, u_m) = \underbrace{0 + \dots + 0}_{m}.$$

Since $U_1 + \dots U_m$ is a direct sum, by 1.44 of Axler, the only way to write the zero vector $0 + \dots + 0$ is to take

$$u_1 = 0, \dots, u_m = 0$$

So $(u_1, \dots, u_m) = 0$, and so $\text{null}\, T = \{0\}$. By 3.16 of Axler, T is injective.

<u>Backward direction</u>: If T is injective, then $U_1 + \dots + U_m$ is a direct sum.

Since T is injective, by 3.16 of Axler, we have

$$\text{null}\, T = \{(\underset{u_1}{0}, \dots, \underset{u_m}{0})\}$$

So the only way to write $0 + \dots + 0$ is to take

$$u_1 = 0, \dots, u_m = 0$$

By 1.44 of Axler, $U_1 + \dots + U_m$ is a direct sum $\boxed{\checkmark}$

<u>3.78 A sum is a direct sum if and only if dimensions add up</u>

Suppose V is finite-dimensional and $U_1, \dots, U_m$ are subspaces of V. then $U_1 + \dots + U_m$ is a direct sum if and only if

$$\dim(U_1 + \dots + U_m) = \dim U_1 + \dots + \dim U_m$$

Proof: By the proof of 3.77 of Axler, the map

$$\Gamma : V_1 \times \dots \times U_m \to U_1 + \dots + U_m \text{ defined by}$$

$$\Gamma(u_1, \dots, u_m) = u_1 + \dots + u_m$$

is surjective. So the Fundamental Theorem of Linear Maps (3.22 of Axler) gives us

$$\begin{aligned}
\dim(U_1 + \dots + U_m) &= \dim \text{range}\, \Gamma && \Gamma \text{ is surjective: range } \Gamma = U_1 + \dots + U_m \\
&= \dim(U_1 \times \dots \times U_m) - \dim \text{null}\, \Gamma && \text{by Fund. thm of Linear maps} \\
&= \dim(U_1 \times \dots \times U_m) - \dim\{0\} && \text{iff } \Gamma \text{ is injective (3.16 of Axler)} \\
&= \dim(U_1 \times \dots \times U_m)
\end{aligned}$$

if and only if Γ is injective

Combine with 3.77 and 3.76 of Axler to conclude that $U_1 + \dots + U_m$ is a direct sum if and only if we have

$$\begin{aligned}
\dim(U_1 + \dots + U_m) &= \dim(U_1 \times \dots \times U_m) \\
&= \dim U_1 + \dots + \dim U_m && \text{by 3.76 of Axler}
\end{aligned}$$

07-22-19

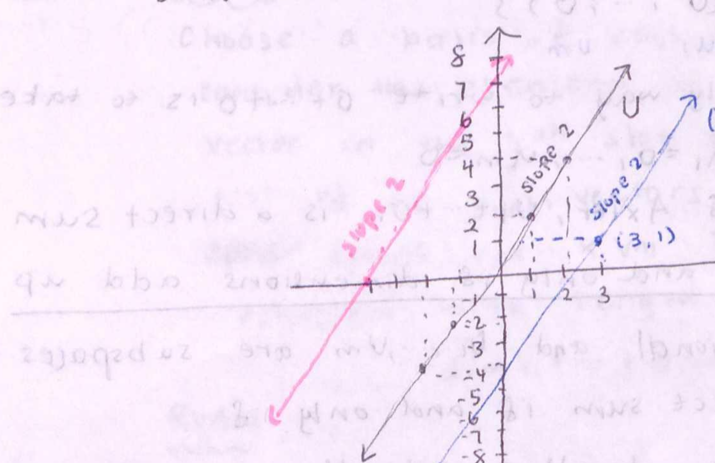
Quotients of vector spaces

3.79 Definition

Suppose $v \in V$ and U is a subspace of V.
Then $v+U$ is the subset of V defined
$$v+U = \{v+u : u \in U\}$$

3.80 Example

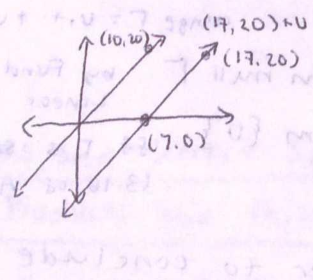
Let $V = \mathbb{R}^2$ and $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$
Then U is the line in $\mathbb{R}^2$ through the origin with slope 2



So $(3,1)+U$ is a line in $\mathbb{R}^2$ that contains the point $(3,1)$ and has a slope 2 and $(-4,0)+U$ is a line in $\mathbb{R}^2$ that contains the point $(-4,0)$ and has slope 2.

$$(3,1)+U = \{(3,1)+(x,2x) : x \in \mathbb{R}\}$$
$$= \{(3+x, 1+2x) : x \in \mathbb{R}\}$$

$$(-4,0)+U = \{(-4,0)+(x,2x) : x \in \mathbb{R}\}$$
$$= \{(-4+x, 2x) : x \in \mathbb{R}\}$$



Prove: Since $(7,0)$ and $(17,20)$ lie on the same line,
$$(7,0)+U = (17,20)+U$$

Proof: $(17,20)+U = \{(17,20)+(x,2x) : (x,2x) \in U\}$
$$= \{(17+x, 20+2x) : x \in \mathbb{R}\}$$

$$\boxed{(7,0)+U} = \{(7,0)+(x,2x) : (x,2x) \in U\}$$
$$= \{(7+x, 2x) : x \in \mathbb{R}\}$$
$$= \{(17-10+x, 20-20+2x) : x \in \mathbb{R}\}$$
$$= \{(17+(x-10), 20+2(x-10)) : x \in \mathbb{R}\}$$
$$= \{(17+y, 20+2y) : y \in \mathbb{R}\}$$
$$= \{(17,20)+(y,2y) : y \in \mathbb{R}\}$$
$$= \boxed{(17,20)+U}$$

Let $y = x-10$
Since $x \in \mathbb{R}$, it follows the $y \in \mathbb{R}$.

3.81 Definition

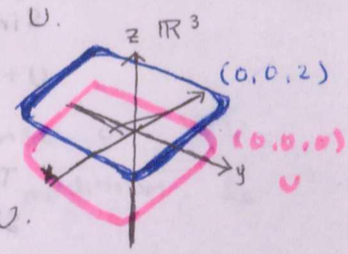
- A affine subset of V is a subset of V of the form $v+U$ for some $v \in V$ and some subspace U of V
- If U is a subspace of V, for all $v \in V$, the affine subset $v+U$ is said to be parallel to U.

3.82 Example

- Let $V = \mathbb{R}^2$ and $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ as in Example 3.80
  Then all the lines in $\mathbb{R}^2$ with slope 2 are parallel to U. And these lines are affine subsets in $\mathbb{R}^2$
- Let $V = \mathbb{R}^3$ and $U = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$. Then the affine subsets of $\mathbb{R}^3$ are all the planes in $\mathbb{R}^3$ that are parallel to U.
  For example,

$$(0,0,2)+U = \{(0,0,2)+(x,y,0) : x, y \in \mathbb{R}\}$$
$$= \{(x,y,2) : x, y \in \mathbb{R}\}$$

is an affine subset of $\mathbb{R}^3$ and is parallel to U.



3.83 Definition

Let U be a subspace of V. Then the quotient space V/U is the set of all affine subsets of V parallel to U:

$$V/U = \{v+U : v \in V\}$$

quotient space

eg: $(7,0)+U$ is an affine subset of $\mathbb{R}^2$
$(7,0)+U \in \mathbb{R}^2/U$

3.84 Example

- If $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, then $\mathbb{R}^2/U$ is the set of all lines in $\mathbb{R}^2$ that have slope 2
- If U is a line in $\mathbb{R}^3$ containing the origin, then $\mathbb{R}^3/U$ is the set of all lines in $\mathbb{R}^3$ parallel to U

For example,

$$U_1 = \{(x,y,0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$$
$$\mathbb{R}^3/U_1 = \{(0,0,z)+U_1 : x, y, z \in \mathbb{R}\}$$
$$U_2 = \{(0,y,z) \in \mathbb{R}^3 : x, y \in \mathbb{R}\},$$
$$\mathbb{R}^3/U_2 = \{(x,0,0)+U_2 : x, y, z \in \mathbb{R}\}$$

IMPORTANT RESULT FOR UPCOMING EXAMS

3.85 Two affine subsets parallel to U are equal or disjoint

Let U be a subspace of V and $v, w \in V$
Then the following are equivalent

(a) $v - w \in U$;
(b) $v+U = w+U$
(c) $(v+U) \cap (w+U) \neq \emptyset$

Proof : (a) implies (b):

Suppose (a) holds: $v-w \in U$

Let $u \in U$ be arbitrary

Since U is a subspace of V, in particular it is closed under addition.

Sine $u \in U$ and $v-w \in U$, we have $\underline{(v-w)+u \in U}$.

For all $u \in U$, we have

$$\begin{aligned} v+u &= w+v-w-u \\ &= w+\underline{((v-w)+u)} \in U \\ &\in w+U \end{aligned}$$

Therefore $v+U \subset w+U$

Similarly, for all $u \in U$ we have

$$\begin{aligned} w+u &= v+w-v+u \\ &= v+\underline{(-(v-w)+u)} \in U \\ &\in v+U \end{aligned}$$

* Therefore $w+U \subset v+U$

So we conclude the set equality

$$v+U = w+U,$$

which is (b).

(b) implies (c)

Suppose (b) holds: $v+U = w+U$

Then there exists $u \in U$ that satisfies

$$\begin{aligned} v+u &\in v+U \\ &= w+U \end{aligned}$$

So $v+u \in v+U$ and $v+u \in w+U$

That is, $v+u \in (v+U) \cap (w+U)$

In other words

$$(v+U) \cap (w+U) \neq \emptyset.$$

which is (c)

(c) implies (a)

Suppose (c) holds: $(v+U) \cap (w+U) \neq \emptyset$. Then there exist $u_1, u_2 \in U$ that satisfies

$$\underbrace{v+u_1}_{\in v+U} = \underbrace{w+u_2}_{\in w+U}$$

Since U is a subspace of V, it is closed under addition and scalar multiplication, which means $u_1-u_2 \in U$. In fact, we have

$$\begin{aligned} v-w &= u_2+u_1 \\ &= -(u_1-u_2) \\ &\in U \end{aligned}$$

which is (a)

3.86 Definition

Let U be a subspace of V. Then:

- **addition** is defined on V/U by
$$(v+U)+(w+U) = (v+w)+U$$
- **scalar multiplication** is defined on V/U by
$$\lambda(v+U) = (\lambda v)+U$$

3.87 Quotient space is a vector space

Let U be a subspace of V. Then V/U is a vector space with respect to the operations defined in Definition 3.86.

Proof: Let $v, w \in V$ be arbitrary

First, we need to show that the operations of addition and scalar multiplication make sense on V/U.

Suppose $\hat{v}, \hat{w} \in V$ satisfy $v+U = \hat{v}+U$ and $w+U = \hat{w}+U$.

First, we will show that addition makes sense on V/U.

since U is a subspace of V, it is closed under addition. So
$$(v+w)-(\hat{v}+\hat{w}) = v-\hat{v}+w-\hat{w} \in U$$

By 3.85 of Axler
$$(v+w)+U = (\hat{v}+\hat{w})+U$$

So addition makes sense on V/U

Now let $\lambda \in \mathbb{F}$ be arbitrary

suppose $\hat{v} \in V$ satisfies $v+U = \hat{v}+U$

By 3.85 of Axler, $v-\hat{v} \in U$. Since U is a subspace of V, it is closed under scalar multiplication, which means

$\lambda(u-\hat{u}) \in U$

So we have
$$\lambda v - \lambda\hat{v} = \lambda(v-\hat{v}) \in U$$

By 3.85 Axler
$$\lambda v + U = \lambda\hat{v}+U$$

So scalar multiplication makes sense on V/U

Next we will show that V/U satisfies all axioms of a vector space

Let $v, w, x \in V$ and $\lambda \in \mathbb{F}$ (a, b) be arbitrary

- Commutativity: $(v+U)+(w+U) = (v+w)+U$
$$= (w+v)+U$$
$$= (w+U)+(v+U)$$
- Associativity: $((v+U)+(w+U))+(x+U) = ((v+w)+U)+(x+U)$
$$= ((v+w)+x)+U$$
$$= (v+(w+x))+U$$
$$= (v+U)+((w+x)+U)$$
$$= (v+U)+((w+U)+(x+U))$$

• Additive identity : $(v+U)+(0+U) = (v+0)+U$
$= v+U$

• Additive inverse : $(v+U)+((-v)+U) = (v+(-v))+U$
$= 0+U$

• Multiplicative identity : $1(v+U) = (1v)+U$
$= v+U$

• Distributive properties : $a((v+U)+(w+U)) = a((v+w)+U)$
$= a(v+w)+U$
$= (av+aw)+U$
$= ((av)+U)+((aw)+U)$
$= a(v+U)+a(w+U)$

and $(a+b)(v+U) = ((a+b)v)+U$
$= (av+bv)+U$
$= ((av)+U)+((bv)+U)$
$= a(v+U)+b(v+U)$ ∎

3.88 Definition

Let U be a subspace of V. The quotient map is the linear map $\pi : V \to V/U$ defined by
$$\pi(v) = v+U$$
for all $v \in V$

3.89 Dimension of a quotient space

Suppose V is finite dimensional, and U is a subspace of V. Then
$$\dim V/U = \dim V - \dim U$$

Proof: Let $\pi : V \to V/U$ be the quotient map.

First, we claim null $\pi = U$
Since $v \in U$, we have $v - 0 = v \in U$, so by 3.85 of Axler,
$$v+U = 0+U$$
In fact, we have
$$\pi(v) = v+U$$
$$= 0+U$$
So $v \in \text{null } \pi$, and so $U \subset \text{null } \pi$

If $v \in \text{null } \pi$, then $\pi(v) = 0+U$.
Since we also have $\pi(v) = v+U$.
we conclude
$$v+U = 0+U$$
By 3.85 of Axler,
$$v = v - 0 \in U$$
So null $\pi \subset U$.

Therefore, we conclude the set equality
$$\text{null } \pi = U$$

Next claim : range $\pi = V/U$ *
(will come back to this tomorrow)

By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\dim V = \dim \operatorname{null} \pi + \dim \operatorname{range} \pi$$
$$= \dim U + \dim V/U$$

as desired.

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range $\pi = V/U$

Let $w \in$ range π.
Then $w = \pi(v)$ for some $v \in U$
In fact, by Definition 3.88, we have

$$w = \pi(v)$$
$$= v + U$$
$$\in V/U$$

So we get range $\pi \subset V/U$

Suppose we have

$$v + U \in V/U$$

By Definition 3.88,

$$v + U = \pi(v) \in \text{range } \pi$$

So $V/U \subset$ range π

Therefore range $\pi = V/U$ ▨

3.90 Definition

Suppose $T \in \mathcal{L}(V, W)$. Define $\tilde{T} : V/(\operatorname{null} T) \to W$ by $\tilde{T}(v + \operatorname{null} T) = Tv$.

Show that $\tilde{T}$ makes sense ($\tilde{T}$ is well-defined)

Suppose $u, v \in V$ satisfy

$$u + \operatorname{null} T = v + \operatorname{null} T$$

By 3.85 of Axler, we have

$$u - v \in \operatorname{null} T$$

This means

$$T(u - v) = 0$$

In fact, we have

$$Tu - Tv = T(u - v) = 0,$$

So $Tu = Tv$
Therefore, $\tilde{T}(u + \operatorname{null} T) = Tu$
$= Tv = \tilde{T}(v + \operatorname{null} T)$,
and so $\tilde{T}$ is well-defined.

3.91 Null Space and range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V, W)$ Then:
(a) $\tilde{T} : V/(\operatorname{null} T) \to W$ is a linear map; $\tilde{T} \in \mathcal{L}(V/(\operatorname{null} T), W)$
(b) $\tilde{T}$ is injective
(c) range $\tilde{T}$ = range T
(d) $V/(\operatorname{null} T)$ is isomorphic to range T

Proof: (a) Let $u, v \in V$ and $\lambda \in \mathbb{F}$

- Additivity: $\tilde{T}((u+\text{null}\, T)+(v+\text{null}\, T))$
$= \tilde{T}((u+v)+\text{null}\, T)$
$= T(u+v)$
$= Tu + Tv$
$= \tilde{T}(u+\text{null}\, T) + \tilde{T}(v+\text{null}\, T)$

- Homogeneity: $\tilde{T}(\lambda(v+\text{null}\, T)) = \tilde{T}(\lambda v + \text{null}\, T)$
$= T(\lambda v)$
$= \lambda T v$
$= \lambda \tilde{T}(v+\text{null}\, T)$

Therefore, $\tilde{T}$ is linear.

will come back to this

(b) Suppose $v \in V$ and $\tilde{T}(v+\text{null}\, T) = 0$

Then we have

$Tv = \tilde{T}(v+\text{null}\, T)$
$= 0$

So $v - 0 = v \in \text{null}\, T$

By 3.85 of Axler

$v+\text{null}\, T = 0 + \text{null}\, T$

By part (a), $\tilde{T}$ is linear. So (3.11 of Axler) we have

$\tilde{T}(0+\text{null}\, T) = 0$

So $0 + \text{null}\, T \in \text{null}\, \tilde{T}$ or $\{0+\text{null}\, T\} \subset \text{null}\, \tilde{T}$

So $\tilde{T}(0+\text{null}\, T) = \tilde{T}(v+\text{null}\, T)$
$= 0$

(c) For all $v \in V$, $\tilde{T}(v+\text{null}\, T) = Tv$

Suppose $w \in \text{range}\, T$. Then $w = Tv$ for some $v \in V$. In fact,

$w = Tv$
$= \tilde{T}(v+\text{null}\, T) \in \text{range}\, \tilde{T}$

So $\text{range}\, T \subset \text{range}\, \tilde{T}$

Suppose $x \in \text{range}\, \tilde{T}$. Then $x = \tilde{T}(v+\text{null}\, T)$ for some $v \in V$

In fact,

$x = \tilde{T}(v+\text{null}\, T)$
$= Tv \in \text{range}\, T$

So $\text{range}\, \tilde{T} \subset \text{range}\, T$.

Therefore we conclude $\text{range}\, \tilde{T} = \text{range}\, T$

(d): By part (c) range $\tilde{T}$ = range T

If we think of $\tilde{T}$ as a map into range $\tilde{T}$

$\tilde{T}: V/(\text{null } T) \to \text{range } T$ is surjective

So $\tilde{T}$ is also surjective

By part (b), $\tilde{T}$ is also injective

Therefore, by 3.56 of Axler, $\tilde{T}$ is invertible

By part (a), T is linear.

Therefore $\tilde{T}: V/(\text{null } T) \to \text{range } T$ is an isomorphism

(b): Suppose $v \in V$ satisfies $\tilde{T}(v + \text{null } T) = 0$.

Then

$$Tv = \tilde{T}(v + \text{null } T) = 0$$

So $v - 0 = v \in \text{null } T$. By 3.85 of Axler,

$$v + \text{null } T = 0 + \text{null } T$$

additive identity of $V/(\text{null } T)$

Therefore, $\text{null } \tilde{T} \subset \{0 + \text{null } T\}$

But $\tilde{T}(0 + \text{null } T) = 0$ Since $\tilde{T}$ is linear

So $\{0 + \text{null } T\} \subset \text{null } \tilde{T}$

So $\text{null } \tilde{T} = \{0 + \text{null } T\}$

By 3.16 of Axler, $\tilde{T}$ is injective.

$\blacksquare$