3.E Products and Quotients of Vector Spaces

3.71 Definition

Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbb{F}$.

- The product $V_1 \times \cdots \times V_m$ is defined by

$$V_1 \times \cdots \times V_m = \{(v_1, \ldots, v_m) : v_1 \in V_1, \ldots, v_m \in V_m\}.$$

- Addition on $V_1 \times \cdots \times V_m$ is defined by

$$(u_1, \ldots, u_m) + (v_1, \ldots, v_m) = (u_1 + v_1, \ldots, u_m + v_m).$$

- Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by

$$\lambda(v_1, \ldots, v_m) = (\lambda v_1, \ldots, \lambda v_m).$$

3.72 Example

Example : $\underbrace{(5-6x+4x^2, (3, 8, 7))}_{\text{Length } 3} \in P_2(\mathbb{R}) \times \mathbb{R}^3.$

Example : $\underbrace{((1,2), (3,4,5))}_{\text{Length } 2} \in \mathbb{R}^2 \times \mathbb{R}^3$

Example : $\underbrace{(1, (2,3), (4,5))}_{\text{Length } 3} \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$

3.73 Product of vector spaces is a vector space

Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbb{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

Proof : Let $u_i, v_i, w_i \in V_i$ for each $i = 1, \ldots, m$, and let $\lambda \in \mathbb{F}$.

- Commutativity: Since each V_i is a vector space, we have $u_i+v_i=v_i+u_i$

  So we have

$$\begin{aligned}(u_1,\ldots,u_m)+(v_1,\ldots,v_m) &= (u_1+v_1,\ldots,u_m+v_m)\\ &= (v_1+u_1,\ldots,v_m+u_m)\\ &= (v_1,\ldots,v_m)+(u_1,\ldots,u_m)\end{aligned}$$

- associativity: Since V_i is a vector space, we have $(u_i+v_i)+w_i = u_i+(v_i+w_i)$

  So we have

$$\begin{aligned}((u_1,\ldots,u_m)+(v_1,\ldots,v_m) &= (u_1+v_1,\ldots,u_m+v_m)+(w_1,\ldots,w_m)\\ &= ((u_1+v_1)+w_1,\ldots,(u_m+v_m)+w_m)\\ &= (u_1+(v_1+w_1),\ldots,u_m+(v_m+w_m))\\ &= (u_1,\ldots,u_m)+(v_1+w_1,\ldots,v_m+w_m)\\ &= (u_1,\ldots,u_m)+((v_1,\ldots,v_m)+(w_1,\ldots,w_m))\end{aligned}$$

- Additive identity: We have $(0,\ldots,0)\in V_1\times\cdots\times V_m$. And it satisfies

$$\begin{aligned}(v_1,\ldots,v_m)+(0,\ldots,0) &= (v_1+0,\ldots,v_m+0)\\ &= (v_1,\ldots,v_m)\end{aligned}$$

  So $(0,\ldots,0)$ is the additive identity of $V_1\times\cdots\times V_m$.

- Additive inverse We have $(-v_1,\ldots,-v_m)\in V_1\times\cdots\times V_m$. And it satisfies

$$\begin{aligned}(v_1,\ldots,v_m)+(-v_1,\ldots,-v_m) &= (v_1+(-v_1),\ldots,v_m+(-v_m))\\ &= (v_1-v_1,\ldots,v_m-v_m)\\ &= (0,\ldots,0)\end{aligned}$$

  So $(-v_1,\ldots,-v_m)$ is the additive inverse of $V_1\times\cdots V_m$.

- Multiplicative identity We have

$$\begin{aligned}1(v_1,\ldots,v_m) &= (1v_1,\ldots,1v_m)\\ &= (v_1,\ldots,v_m)\end{aligned}$$

• Distributive properties

For all $a, b \in \mathbb{F}$, we have

$$a((u_1, \ldots, u_m) + (v_1, \ldots, v_m)) = a(u_1+v_1, \ldots, u_m+v_m)$$
$$= (a(u_1+v_1), \ldots, a(u_m+v_m))$$
$$= (au_1 + av_1, \ldots, au_m + av_m)$$
$$= (au_1, \ldots, au_m) + (av_1, \ldots, av_m)$$
$$= a(u_1, \ldots, u_m) + a(v_1, \ldots, v_m)$$

and $(a+b)(v_1, \ldots, v_m) = ((a+b)v_1, \ldots, (a+b)v_m)$

$$= (av_1, \ldots, av_m) + (bv_1, \ldots, bv_m)$$
$$= a(v_1, \ldots, v_m) + b(v_1, \ldots, v_m)$$

3.74 Example Show that $\mathbb{R}^2 \times \mathbb{R}^3$ is isomorphic to $\mathbb{R}^5$.

Note that, as vector spaces, $\mathbb{R}^2 \times \mathbb{R}^3 \neq \mathbb{R}^5$ because ~~element~~ elements $((x_1, x_2), (x_3, x_4, x_5))$ of $\mathbb{R}^2 \times \mathbb{R}^3$ have length 2 (length 2) but elements $\underbrace{(x_1, x_2, x_3, x_4, x_5)}_{\text{length } 5}$ of $\mathbb{R}^5$ have length 5.

Proof: Define $T: \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$ by $T((x_1, x_2), (x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4, x_5)$.

First, we will show that T is injective.

Let $((x_1, x_2), (x_3, x_4, x_5)) \in \operatorname{null} T$, which means

$$T((x_1, x_2), (x_3, x_4, x_5)) = (0, 0, 0, 0, 0)$$

Then we have $(0,0,0,0,0) = T((x_1, x_2), (x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4, x_5)$.

So $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$.

This means $((x_1, x_2), (x_3, x_4, x_5)) = ((0,0), (0,0,0))$

So null $T \subset \{((0,0),(0,0,0))\}$.

$$T(((0,0),(0,0,0))) = (0,0,0,0,0)$$

we also have $\{((0,0),(0,0,0))\} \subset$ null T

Therefore, null $T = \{((0,0),(0,0,0,0,0))\}$

By 3.16 of Axler, T is injective.

Next, we will show that T is surjective

For all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, we have

$(x_1, x_2, x_3, x_4, x_5) = T((x_1, x_2), (x_3, x_4, x_5))$

So $\mathbb{R}^5 \subset$ range T. But range T is a subspace of $\mathbb{R}^5$. So we have

$$\text{range } T = \mathbb{R}^5.$$

So T is surjective.

Therefore, by 3.56 of Axler, T is invertible.

Next, we will show that T is linear.

- Additivity For all $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}$,

we have $T((x_1, x_2), (x_3, x_4, x_5)) + ((y_1, y_2), (y_3, y_4, y_5))$

$= T(((x_1+y_1, x_2+y_2), (x_3+y_3, x_4+y_4, x_5+y_5)))$

$= (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5)$

$= (x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)$

$= T((x_1, x_2), (x_3, x_4, x_5)) + T((y_1, y_2), (y_3, y_4, y_5))$

- Homogeneity : For all $\lambda \in \mathbb{F}$ and for all $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$,

we have $T(\lambda((x_1, x_2)(x_3, x_4, x_5))) = T(((\lambda x_1, \lambda x_2); (\lambda x_3, \lambda x_4, \lambda x_5)))$

$= (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5)$

$= \lambda(x_1, x_2, x_3, x_4, x_5)$

$= \lambda T(((x_1, x_2), (x_3, x_4, x_5)))$

Therefore, T is linear.

So T is invertible and linear.

Therefor T is an isomorphism.

3.75 Example

Find a basis of $P_2(\mathbb{R}) \times \mathbb{R}^2$

solution: $(1, (0,0)), (x, (0,0)), (x^2, (0,0)), (0, (1,0)), (0, (0,1))$

$1, x, x^2$ is a basis of $P_2(\mathbb{R})$

$(1,0), (0,1)$ is a basis of $\mathbb{R}^2$.

3.76 Dimension of a product is the sum of dimensions

Suppose $V_1, \ldots, V_m$ are finite-dimensional vector spaces.
Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m.$$

Proof: Axler

Choose a basis of each V_j. For each basis vector of each V_j consider the element of $V_1 \times \cdots \times V_m$ that equals the basis vector in the j^{th} slot and 0 in the other slots. These list of all such vectors is linearly independent and spans $V_1 \times \cdots \times V_m$. Therefore, it is a basis of $V_1 \times \cdots \times V_m$, with length $\dim V_1 + \cdots + \dim V_m$.

Ayan's interpretation:

Let $j = 1, \ldots, m$. Let $v_{j,1}, \ldots, v_{j,n_j}$ be a basis of each V_j. Then $n_j = \dim V_j$. And the i^{th} basis vector of V_j is v_{j,i_j} for $i = 1, \ldots, n_j$. So we have ?

$(v_{1,1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, v_{1,n})$, length: $n_1 = \dim V_1$

$(v_{2,1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, v_{2,n})$, length: $n_2 = \dim V_2$

$\vdots$

$(v_{m,1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, v_{m,n_m})$, length $n_m = \dim V_m$

Total length: $n_1 + n_2 + \cdots + n_m = \dim V_1 + \dim V_2 + \cdots + \dim V_m$.

is a basis of $V_1 \times \cdots \times V_m$.

Products and Direct Sums

3.77 Product and Direct Sums

Suppose that $V_1, \dots, V_m$ are subspaces of V.

Define a linear map

$$\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m \text{ by}$$

$$\Gamma(u_1, \dots, u_m) = u_1 + \cdots + u_m.$$

Then $U_1 + \cdots + U_m$ is a direct sum if and only if Γ is injective.

Proof: Forward direction: If $U_1 + \cdots + U_m$ is a direct sum, then Γ is injective.

Suppose $(u_1, \dots, u_m) \in \text{null}\, \Gamma$, so that $\Gamma(u_1, \dots, u_m) = \underbrace{0 + \cdots + 0}_{m}$.

Since $U_1 + \cdots + U_m$ is a direct sums by 1.44 of Axler, the only way to write the zero vector $0 + \cdots + 0$ is to take $u_1 = 0, \dots, u_m = 0$,

So $(u_1, \dots, u_m) = 0$, and so null $\Gamma = \{0\}$. By 3.16 of Axler, Γ is injective.

Backward direction: If Γ is injective, then $U_1 + \cdots + U_m$ is a direct sum.

Since Γ is injective, by 3.16 of Axler, we have $\text{null}\,\Gamma = \{(\overset{u_1}{0}, \dots, \overset{u_m}{0})\}$

So the only way to write $0 + \cdots + 0$ is to take $u_1 = 0, \dots, u_m = 0$.

By 1.44 of Axler, $U_1 + \cdots + U_m$ is a direct sum.

3.78 A sum is a direct sum if and only if dimensions add up

Suppose V is finite-dimensional and $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if $\dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m$.

Proof: By the proof of 3.77 of Axler, the map $\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$ defined by $\Gamma(u_1, \dots, u_m) = u_1 + \cdots + u_m$. is surjective

So the Fundamental Theorem of Linear Map (3.22 of Axler) gives us

$$\dim(U_1+\cdots+U_m) = \dim \operatorname{range} T$$ (because T is surjective, $\operatorname{range} T = U_1+\cdots+U_m$)

$$= \dim(U_1\times\cdots\times U_m) - \dim \operatorname{null} T$$ by Fund. Thm. of Linear Maps

$$= \dim(U_1\times\cdots\times U_m) - \dim\{0\}$$ if and only if T is injective (3.16 of Axler)

$$= \dim(U_1\times\cdots\times U_m)$$

Combine with 3.77 and 3.76 of Axler to conclude that $U_1+\cdots+U_m$ is a direct sum if and only if we have $\dim(U_1+\cdots+U_m) = \dim(U_1\times\cdots\times U_m)$

$= \dim U_1 + \cdots + \dim U_m$ ~~By 3.76 of Ax~~ by 3.76 of Axler.

3.E Continued

Quotients of ~~Vect~~ Vector Spaces

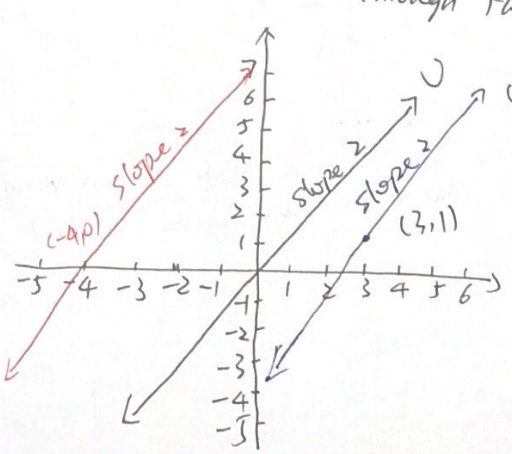
3.79 Definition

Suppose $v\in V$ and U is a subspace of V. Then $v+U$ is the subset of V defined

$$v+U = \{v+u,\ u\in U\}.$$

3.80 Example

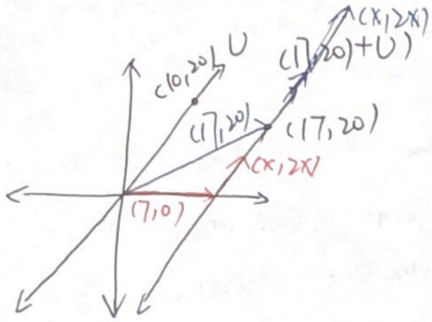
Let $V=\mathbb{R}^2$ and $U=\{(x, 2x)\in\mathbb{R}^2 : x\in\mathbb{R}\}$.
Then U is the line in $\mathbb{R}^2$ through the origin with slope 2.



So $(3,1)+U$ is a line in $\mathbb{R}^2$ that contains the point $(3,1)$ and has slope 2 and $(-4,0)+U$ is a line in $\mathbb{R}^2$ that contains the point $(-4,0)$ and has slope 2.

$(3,1)+U = \{(3,1)+(x,2x) : x \in \mathbb{R}\} = \{(3+x, 1+2x) : x \in \mathbb{R}\}$.

$(-4,0)+U = \{(-4,0)+(x,2x) : x \in \mathbb{R}\} = \{(-4+x, 2x) : x \in \mathbb{R}\}$.



Prove: Since $(7,0)$ and $(17,20)$ lie on the same line, $(7,0)+U = (17,20)+U$.

<u>Proof</u>: $(17,20)+U = \{(17,20)+(x,2x) : (x,2x) \in U\}$
$= \{(17+x, 20+2x) : x \in \mathbb{R}\}$.

$\boxed{(7,0)+U} = \{(7,0)+(x,2x) : (x,2x) \in U\}$
$= \{(7+x, 2x) : x \in \mathbb{R}\}$.
$= \{(17-10+x, 20-20+2x) : x \in \mathbb{R}\}$.
$= \{(17+(x-10), 20+2(x-10)) : x \in \mathbb{R}\}$.
$= \{(17+y, 20+2y), y \in \mathbb{R}\}$.
$= \{(17,20)+(y,2y) : y \in \mathbb{R}\}$.
$= \boxed{(17,20)+U}$

Let $y = x-10$
Since $x \in \mathbb{R}$,
it follows that
$y \in \mathbb{R}$.

Similarly,

$(8,2)+U = (17,20)+U$
$(8,2)+U = (7,0)+U$
$(8,4)+U \neq (17,20)+U$

<u>3.81 Definition</u>

- An <u>affine subset</u> of V is a subset of V of the form $v+U$ for some $v \in V$ and some subspace U of V.
- If U is a subspace of V, for all $v \in V$, the affine subset $v+U$ is said to be <u>parallel</u> to U.

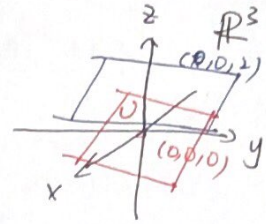
<u>3.82 Example</u>

- Let $V = \mathbb{R}^2$ and $U = \{(x,2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, as in Example 3.80. Then all the lines in $\mathbb{R}^2$ with slope 2 are parallel to U. And these lines are affine subsets in $\mathbb{R}^2$.
- Let $V = \mathbb{R}^3$ and $U = \{(x,y,0) \in \mathbb{R}^3 : x,y \in \mathbb{R}\}$.

Then the affine subset of $\mathbb{R}^3$ are all the planes in $\mathbb{R}^3$ that are parallel to U.

For example, $(0,0,2)+U=\{(0,0,2)+(x,y,0): x,y\in\mathbb{R}\}$.
$= \{(x,y,2): x,y\in\mathbb{R}\}$.

is an affine subset of $\mathbb{R}^3$ and is parallel to U.



3.83 Definition

Let U be a subspace of V. Then the quotient space V/U is the set of all affine subsets of V parallel to U, written:

quotient space

$$V/U = \{v+U : v\in V\}.$$

Example: $(7,0)+U$ is an affine subset of $\mathbb{R}^2$
$(7,0)+U\in\mathbb{R}^2/U$.

3.84 Example

- If $U=\{(x,2x)\in\mathbb{R}^2: x\in\mathbb{R}\}$, then $\mathbb{R}^2/U$ is the set of all lines in $\mathbb{R}$ that have slope 2.
- If U is a line in $\mathbb{R}^3$ containing the origin, then $\mathbb{R}^3/U$ is the set of all lines in $\mathbb{R}^2$ parallel to U.

For example, $U_1=\{(x,y,0)\in\mathbb{R}^3 : x,y\in\mathbb{R}\}$.

$\mathbb{R}^3/U_1 = \{(0,0,z)+U_1 : x,y,z\in\mathbb{R}\}$.

$U_2=\{(0,y,z)\in\mathbb{R}^2: x,y\in\mathbb{R}\}$.

$\mathbb{R}^3/U_2=\{(x,0,0)+U_2 : x,y,z\in\mathbb{R}\}$.

Important result for upcoming Exams !!!

3.85 Two affine subsets parallel to U are equal or disjoint.

Let U be a subspace of V and $v,w\in V$.
Then the following are equivalent:

(a) $v-w\in U$.

(b) $v+U=w+U$

(c) $(v+U)\cap(w+U)\neq\phi$

Proof: (a) implies (b):

Suppose (a) holds: $v-w \in U$. Let $u \in U$ be arbitrary.
Since U is a subspace of V, in particular it is closed under addition. Since $u \in U$ and $v-w \in U$, we have $(v-w)+u \in U$.
For all $u \in U$, we have

$$\begin{aligned} v+u &= w+v-w+u \quad \in U \\ &= w+((v-w)+u) \\ &\in w+U \end{aligned}$$

Similarly, for all $u \in U$, we have

$$\begin{aligned} w+u &= v+w-v+u \\ &= v+(-(v-w)+u) \quad \in U \\ &\in v+U \end{aligned}$$

Therefore, $w+U \subset v+U$.

So we conclude the set equility $v+U = w+U$, which is (b).

(b) implies (c)

Suppose (b) holds: $v+U = w+U$. Then there exists $u \in U$ that satisfies

$$\begin{aligned} v+u &\in v+U \\ &= w+U. \end{aligned}$$

So $v+u \in v+U$ and $v+u \in w+U$.
That is, $v+u \in (v+U) \cap (w+U)$.
In other words, $(v+U) \cap (w+U) \neq \emptyset$, which is (c).

(c) implies (a)

Suppose (c) holds: $(v+U) \cap (w+U) \neq \emptyset$. Then there exist $u_1, u_2 \in U$ that satisfies

$$\overset{\in v+U}{v+u_1} = \overset{w+U}{w+u_2}$$

Since U is a subspace of V, it is closed under addition and scalar multiplication, which means $u_1 - u_2 \in U$. In fact, we have

$$\begin{aligned} v-w &= u_2 - u_1 \\ &= -(u_1 - u_2) \\ &\in U, \end{aligned}$$

Which is (a).

3.86 Definition

Let U be a subspace of V. Then:

- addition is defined on V/U by $(v+U)+(w+U)=(v+w)+U$.
- scalar multiplication is defined on V/U by $\lambda(v+U)=(\lambda v)+U$.

3.87 Quotient space is a vector space

Let U be a subspace of V. Then V/U is a vector space with respect to the operators defined in Definition 3.86.

Proof: Let $v, w \in V$ be arbitrary.
First, we need to show that the operators of addition and scalar multiplication make sense on V/U.

Suppose $\hat{v}, \hat{w} \in V$ satisfy $v+U=\hat{v}+U$ and $w+U=\hat{w}+U$.
First, we will show that addition makes sense on V/U.
Since U is a subspace of V, it is closed under addition. So

$$(v+w)-(\hat{v}+\hat{w})=v-\hat{v}+w-\hat{w} \in U.$$

By 3.85 of Axler, $(v+w)+U=(\hat{v}+\hat{w})+U$.
So addition makes sense on V/U.

Now let $\lambda \in \mathbb{F}$ be arbitrary. Suppose $\hat{v} \in V$ satisfies $v+U=\hat{v}+U$.
By 3.85 of Axler, $v-\hat{v} \in U$. Since U is a subspace of V, it is closed under scalar multiplication, which means $\lambda(u-\hat{u}) \in U$.
So we have $\lambda v-\lambda\hat{v}=\lambda(v-\hat{v}) \in U$.
By 3.85 of Axler, $\lambda v+U=\lambda\hat{v}+U$.
So scalar multiplication makes sense on V/U.

Next, we will show that V/U satisfies all axioms of a vector space.
Let $v, w, x \in V$ and $\lambda \in \mathbb{F}$ be arbitrary.

- Commutativity: $(v+U)+(w+U)=(v+w)+U$
$=(w+v)+U$
$=(w+U)+(v+U)$.

- Associativity: $((v+U)+(w+U))+(x+U) = ((v+w)+U)+(x+U)$
$$= ((v+w)+x)+U$$
$$= (v+(w+x))+U$$
$$= (v+U)+((w+x)+U)$$
$$= (v+U)+((w+U)+(x+U)).$$

- Additive identity: $(v+U)+(0+U) = (v+0)+U$
$$= v+U.$$

- Additive inverse: $(v+U)+((-v)+U) = (v+(-v))+U$
$$= 0+U$$

- Multiplicative identity: $1(v+U) = (1v)+U$
$$= v+U$$

- Distributive properties: $a((v+U)+(w+U)) = a((v+w)+U)$
$$= a(v+w)+U$$
$$= (av+aw)+U$$
$$= ((av)+U)+((aw)+U)$$
$$= a(v+U)+a(w+U)$$

and $(a+b)(v+U) = ((a+b)v)+U$
$$= (av+bv)+U$$
$$= ((av)+U)+((bv)+U)$$
$$= a(v+U)+b(v+U)$$

3.88 Definition

Let U be a subspace of V. The <u>quotient map</u> is the linear map $\pi : V \to V/U$ defined by $\pi(v) = v+U$ for all $v \in V$.

3.89 Dimension of a quotient space

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim V/U = \dim V - \dim U$$

Proof: Let $\pi: V \to V/U$ be the quotient map.

First, we claim $\text{null}\, \pi = U$.

Since $v \in U$, we have $v - 0 = v \in U$, so by 3.88 of Axler,

$$v + U = 0 + U.$$

In fact, we have $\pi(v) = v + U$
$= 0 + U$.

So $v \in \text{null}\, \pi$, and so $U \subset \text{null}\, \pi$.

If $v \in \text{null}\, \pi$, then $\pi(v) = 0 + U$.

Since we also have $\pi(v) = v + U$,

we conclude $v + U = 0 + U$.

By 3.85 of Axler, $v = v - 0 \in U$.

So $\text{null}\, \pi \subset U$.

Therefore, we conclude the set equality $\text{null}\, \pi = U$.

Next claim: $\text{range}\, \pi = V/U$. (will come back to this tomorrow).

By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\dim V = \dim \text{null}\, \pi + \dim \text{range}\, \pi$$
$$= \dim U + \dim V/U$$

as desired.

range $\pi = V/U$

Let $w \in \text{range}\, \pi$

Then $w = \pi(v)$ for some $v \in U$

In fact, by definition 3.88,

we have $w = \pi(v)$
$= v + U$
$\in V/U$

So we get $\text{range}\, \pi \subset V/U$

Suppose we have $v + U \in V/U$

By Definition 3.88, $v + U = \pi(v) \in \text{range}\, \pi$.

So $V/U \subset \text{range}\, \pi$.

Therefore, range $\pi = V/U$.

3.90 Definition

Suppose $T \in \mathcal{L}(V,W)$. Define $\tilde{T}: V/(\text{null } T) \longrightarrow W$ by

$$\tilde{T}(v + \text{null } T) = Tv.$$

Show that $\tilde{T}$ makes sense ($\tilde{T}$ is well-defined).

Suppose $u, v \in V$ satisfy $u + \text{null } T = v + \text{null } T$

By 3.85 of Axler, we have $u - v \in \text{null } T$.

This means $T(u-v) = 0$

In fact, we have $Tu - Tv = T(u-v) = 0$,

So $Tu = Tv$

Therefore, $\tilde{T}(u + \text{null } T) = Tu = Tv = \tilde{T}(v + \text{null } T)$,

and so $\tilde{T}$ is well-defined.

3.91 Null space and range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V,W)$. Then:

(a) $\tilde{T}: V/(\text{null } T) \longrightarrow W$ is a linear map; $\tilde{T} \in \mathcal{L}(V/(\text{null } T), W)$

(b) $\tilde{T}$ is injective

(c) range $\tilde{T}$ = range T

(d) $V/(\text{null } T)$ is isomorphic to range T.

__Proof__: (a) Let $u, v \in V$ and $\lambda \in \mathbb{F}$.

- __Additivity__:
$$\begin{aligned} &\tilde{T}((u + \text{null } T) + (v + \text{null } T)) \\ &= \tilde{T}((u+v) + \text{null } T) \\ &= T(u+v) \\ &= Tu + Tv \\ &= \tilde{T}(u + \text{null } T) + \tilde{T}(v + \text{null } T) \end{aligned}$$

• Homogeneity: $\tilde{T}(\lambda(v+\text{null } T)) = \tilde{T}((\lambda v)+\text{null } T)$

$$= T(\lambda v)$$
$$= \lambda Tv$$
$$= \lambda \tilde{T}(v+\text{null } T)$$

Therefore, $\tilde{T}$ is linear.

(b): Suppose $v \in V$ ~~and~~ satisfies $\tilde{T}(v+\text{null } T) = 0$

~~Then we have~~ and $Tv = \tilde{T}(v+\text{null } T)$

$$= 0$$

So $v - 0 = v \in \text{null } T$

Therefore, $\text{null } \tilde{T} \subset \{0+\text{null } T\}$.

By 3.85 of Axler, $v + \text{null } T = 0 + \text{null } T$. additive identity of $V/(\text{null } T)$

~~By part (a), $\tilde{T}$ is linear. So 3.11 of Axler, we have $\tilde{T}(0+\text{null } T)=0$.~~

~~So $0+\text{null } T \in \text{null } \tilde{T}$, or $\{0+\text{null } T\} \subset \text{null } \tilde{T}$.~~

Therefore, $\text{null } \tilde{T} \subset \{0+\text{null } T\}$.

But $\tilde{T}(0+\text{null } T) = 0$ since $\tilde{T}$ in linear.

So $\{0+\text{null } T\} \subset \text{null } \tilde{T}$.

So $\text{null } \tilde{T} = \{0+\text{null } T\}$.

(c) For all $v \in V$, $\tilde{T}(v+\text{null } T) = Tv$.

Suppose $w \in \text{range } T$. Then $w = Tv$ for some $v \in V$. In fact,

$$w = Tv$$
$$= \tilde{T}(v+\text{null } T)$$
$$\in \text{range } \tilde{T}.$$

So $\text{range } T \subset \text{range } \tilde{T}$.

Suppose $x \in \text{range } \tilde{T}$. Then $x = \tilde{T}(v+\text{null } T)$ for some $v \in V$.

In fact, $x = \tilde{T}(v+\text{null } T) = Tv \in \text{range } T$.

So $\text{range } \tilde{T} \subset \text{range } T$

Therefore, we conclude $\text{range } \tilde{T} = \text{range } T$.

(d): By part (c), range $\tilde{T}$ = range T.

If we think of $\tilde{T}$ as a map into range $\tilde{T}$,

$\tilde{T}: V/(\text{null } T) \longrightarrow \text{range } T$ is surjective.

So $\tilde{T}$ is also surjective.

By part (b), $\tilde{T}$ is also injective.

Therefore, by 3.86 of Axler, $\tilde{T}$ is invertible.

By part (a), $\tilde{T}$ is linear.

Therefore, $\tilde{T} = V/(\text{null } T) \longrightarrow \text{range } T$ is an isomorphism.