3F. Duality

3.92 linear functional

$\varphi \in \mathcal{L}(V, \mathbb{F})$ $\quad \varphi: V \to \mathbb{F}$

3.93 ex: • Define $\varphi: \mathbb{R}^3 \to \mathbb{R}$ by $\varphi(x,y,z) = 4x - 5y + 2z$ Then φ is linear functional.

• $\varphi(x_1, \dots, x_n) = c_1 x_1 + \cdots + c_n x_n$

• $\varphi: P(\mathbb{R}) \to \mathbb{R}$ $\quad \varphi(p) = 3p''(5) + 7p(4)$

• $\varphi(p) = \int_0^1 p(x)\,dx$

3.94 dual space V'

$V' = \mathcal{L}(V, \mathbb{F})$ $\quad V'$ is the vector space of all linear functionals on V.

3.95 $\dim V' = \dim V$

3.96 dual basis

if $v_1, \dots, v_n$ is a basis of V, then dual basis of $v_1, \dots, v_n$ is $\varphi_1, \dots, \varphi_n$ of V'

where each φ_j is the linear functional on V

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

3.97 ex: Sol^n: For all $j = 1, \dots, n$

$$\varphi_j(x_1, \dots, x_n) = c_1 x_1 + \cdots + c_n x_n$$

Then $1 = \varphi_1(v_1) = \varphi_1(e_1) = \varphi_1(1, \dots 0) = c_1$

$0 = \varphi_1(e_2) = c_2$

$\vdots$

$0 = \varphi_1(e_n) = c_n$

$$\varphi_1(x_1, \dots, x_n) = \underset{c_1}{1} x_1 + \underset{c_2}{0} + \cdots + \underset{c_n}{0} = x_1$$

$\varphi_2(x_1, \dots, x_n) = x_2$

$\vdots$

$\varphi_n(x_1, \dots, x_n) = x_n$

3.98 Dual basis is a basis of the dual space.

Suppose V is finite-dimensional. Then the dual basis of V is a basis of V'.

Proof. let $v_1, \ldots v_n$ be a basis of V and let $\varphi_1, \ldots \varphi_n$ be a dual basis of $v_1, \ldots v_n$

we will show that $\varphi_1, \ldots \varphi_n$ is linearly independent.

Suppose $a_1, \ldots a_n \in \mathbb{F}$ satisify.

$$a_1 \varphi_1 + \cdots + a_n \varphi_n = 0.$$

Then for any $j = 1, \ldots, n$ we have

$$(a_1 \varphi_1 + \cdots + a_n \varphi_n) v_j = a_1 \varphi_1 (v_j) + \cdots + a_n \varphi_n (v_j)$$

$$= a_1 \varphi (v_j) + \cdots + a_j \varphi_j (v_j) + \cdots + a_n \varphi_n (v_j)$$

$$= a_1 \cdot 0 + \cdots + a_j \cdot 1 + \cdots + 0 = a_j.$$

So $a_j = (a_1 \varphi_1 + \cdots + a_n V_n)(v_j) = 0(v_j) = 0$

So $a_1 = \cdots = a_n = 0$

then $\varphi_1, \ldots \varphi_n$ are linearly indep.

By 3.95, $\dim V' = \dim V = n$

By 2.39 $\varphi_1, \ldots \varphi_n$ is a basis of V'

3.99 if $T \in \mathcal{L}(V, W)$ then dual map of T is $T' \in \mathcal{L}(W', V')$

$T'(\varphi) = \varphi \circ T$ for $\varphi \in W'$

Show $T' \in \mathcal{L}(W', V')$ let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in W'$ be arbitrary.

$$T'(\varphi + \psi) = (\varphi + \psi) \circ T = \varphi \circ T + \psi \circ T = T'(\varphi) + T'(\psi)$$

$$T'(\lambda \varphi) = \lambda \varphi \circ T = \lambda (\varphi \circ T) = \lambda T'(\varphi)$$

3.100 example.

- Define. $D : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$ by $Dp = p'$
  - Define $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{F})$ by

$$\varphi(p) = p(3)$$

Then $D'(\varphi)$ is linear functional on $\mathcal{P}(\mathbb{R})$ given by.

$$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = p'(3).$$

- Define $\varphi(p) = \int_0^1 p$. then $(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = \int_0^1 p' = p(1) - p(0)$

3.101 Algebraic properties of dual maps

- $(S+T)' = S' + T'$ for $S, T \in \mathcal{L}(V, W)$
- $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$ and all $T \in \mathcal{L}(V, W)$
- $(ST)' = T'S'$ for all $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$

Proof: $(ST)'(\varphi) = \varphi \circ (ST) = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'(\varphi)) = (T'S')(\varphi)$

3.102 If U is a subspace of V, the the annihilator of U, denoted U°.

$$U^\circ = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$$

3.104 EX: $U = \text{span}(e_1, e_2) = \{(x_1, x_2, 0, 0, 0) \in \mathbb{R}^5 : x_1, x_2 \in \mathbb{R}\}$

Show that $U^\circ = \text{span}(\varphi_3, \varphi_4, \varphi_5)$

Sol^n: $\varphi_j(x_1, \dots, x_5) = x_j$

Suppose $\varphi \in \text{span}(\varphi_3, \varphi_4, \varphi_5)$

Then $\varphi = c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$ for some $c_3, c_4, c_5 \in \mathbb{R}$

For all $(x_1, x_2, 0, 0, 0)$

$\varphi(x_1, x_2, 0, 0, 0) = c_3\varphi_3(x_1, x_2, 0, 0, 0) + c_4\varphi_4(x_1, x_2, 0, 0, 0) + c_5\varphi_5(x_1, x_2, 0, 0, 0)$

$= 0$

So $\varphi \in U^\circ$ and so $\text{span}(\varphi_3, \varphi_4, \varphi_5) \subset U^\circ$

Suppose $\varphi \in U^\circ$

So there exists some $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$ s.t.

$\varphi = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_5\varphi_5$

Since $e_1 \in U$ and $\varphi \in U^\circ$,

$0 = \varphi(e_1) = (c_1\varphi_1 + \cdots + c_5\varphi_5)(e_1) = c_1$

Similarly $0 = \varphi(e_2) = c_2$

Hence $\varphi = c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \in U^\circ$

Thus $\varphi \in \text{span}(\varphi_3, \varphi_4, \varphi_5)$

which shows $U^\circ \subset \text{span}(\varphi_3, \varphi_4, \varphi_5)$

So, $U^\circ = \text{span}(\varphi_3, \varphi_4, \varphi_5)$

3.105 U^0 is a subspace

Suppose $U \subset V$ Then $U^0 \subset V'$

annihilator $= \{\varphi \in V' : \varphi(u) = 0\}$

dual space = all v.s of $\mathcal{L}(V, \mathbb{F})$

Proof: Additive identity: since $0(u) = 0$ for all $u \in U$.

we have $0 \in U^0$

closed under addition: for $\varphi, \psi \in U^0$ for all $u \in U$.

$$(\varphi + \psi)(u) = \varphi(u) + \psi(u) = 0 + 0 = 0$$

So $\varphi + \psi \in U^0$

closed under scalar $\lambda \in \mathbb{F}$ $\varphi \in U^0$ $u \in U$

$$(\lambda\varphi)(u) = \lambda(\varphi(u)) = \lambda(0) = 0$$

So $\lambda\varphi \in U^0$

$\square$

3.106 Dimension of the Annihilator

$U \subset V$ then $\dim U + \dim U^0 = \dim V$

reminder: $U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$

$V' = \mathcal{L}(V, \mathbb{F})$

Proof: let $i \in \mathcal{L}(U, V)$ be the inclusion map defined by $i(u) = u$ for all $u \in U$.

Now, Show i' is linear.

let $\lambda \in \mathbb{F}$ $\varphi, \psi \in U$ for all $u \in U$

Additivity: $i'(\varphi+\psi)(u) = ((\varphi+\psi)\circ i)(u) = (\varphi+\psi)(i(u)) = (\varphi+\psi)(u) = \varphi(u) + \psi(u)$

$= \varphi(i(u)) + \psi(i(u)) = (\varphi \circ i)(u) + (\psi \circ i)(u) = (i'(\varphi))(u) + (i'(\psi))(u)$

$= (i'(\varphi) + i'(\psi))(u)$

Thus, $i'(\varphi+\psi) = i'(\varphi) + i'(\psi)$

Homogenenity: $\forall u \in U$ $(i'(\lambda\varphi))(u) = \lambda\varphi(i(u)) = \lambda\varphi(u) = \lambda\, i(\varphi(u))$

Next we show $\text{null } i' = U^0$

$\text{null } i' = \{\varphi \in V' : i'(\varphi) = 0\}$

$= \{\varphi \in V' : \varphi \circ i = 0\}$

$= \{\varphi \in V' : (\varphi \circ i)(u) = 0 \ \forall u \in U\}$

$= \{\varphi \in V' : \varphi(u) = 0 \ \forall u \in U\}$

$= U^0$

Now by Fundamental Theorem of Linear Maps. 3.22

$\dim V = \dim V'$ by 3.95

$= \dim \operatorname{range} i' + \dim \operatorname{null} i'$

$= \dim \operatorname{range} i' + \dim U^0$

Now show $\operatorname{range} i' = U$

Suppose we have $\varphi \in U'$ then φ can be extended to linear functional ψ on V

And by def. of i', $i'(\psi) = \varphi$. So $\varphi \in \operatorname{range} i'$, and so $U' \subset \operatorname{range} i'$

But 3.19 of Axler says i' is a subspace of U'.

Therefore, $\operatorname{range} i' = U'$

So $\dim(\operatorname{range} i') = \dim U'$

$= \dim U$ by 3.95 of Axler

Therefore, $\dim U + \dim U^0 = \dim V$

3.107 The null space of T'

Suppose V and W are finite-dim and $T \in \mathcal{L}(V, W)$ Then

(a) $\operatorname{null} T' = (\operatorname{range} T)^0$

(b) $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$

Proof: (a) Suppose $\varphi \in \operatorname{null} T'$, then $T'(\varphi) = 0$. So for all $v \in V$

we have $0 = 0(v)$

$= (T'(\varphi))(v)$

$= (\varphi \circ T)(v)$

$= \varphi(Tv)$

So $\varphi \in (\operatorname{range} T)^0$

So $\operatorname{null} T' \subset (\operatorname{range} T)^0$

Suppose $\varphi \subset (\operatorname{range} T)^0$ Then $\varphi(Tv) = 0$ for all $v \in V$.

$(T'(\varphi))(v) = (\varphi \circ T)(v) = \varphi(Tv) = \varphi(0) = 0$

and so $\varphi \in \operatorname{null} T'$. So, $(\operatorname{range} T)^0 \subset \operatorname{null} T'$

Therefore, $(\operatorname{range} T)^0 = \operatorname{null} T'$

(b) $\dim \operatorname{null} T' = \dim (\operatorname{range} T)^0$

$= \dim W - \dim (\operatorname{range} T)^0$

$= \dim W - (\dim V - \dim \operatorname{null} T)$

$= \dim \operatorname{null} T + \dim W - \dim V$ $\boxed{//}$

3.108 T surjective is equivalent T' injective

$T \in \mathcal{L}(V, W)$ is injective iff T' is surjective.

<u>Proof.</u> $T \in \mathcal{L}(V, W)$ is surjective iff $\operatorname{range} T = W$ by 3.20 of Axler

iff $(\operatorname{range} T)^\circ = \{0\}$

iff $\operatorname{null} T' = \{0\}$ by 3.107 (a) Axler

iff T' is injective by 3.16 of Axler

To prove $\operatorname{range} T = W \Leftrightarrow (\operatorname{range} T)^\circ = \{0\}$

($\Rightarrow$) Suppose $\operatorname{range} T = W$, that is $\dim \operatorname{range} T = \dim W$, So by 3.106 of Axler, $\dim (\operatorname{range} T)^\circ = 0$

Hence $(\operatorname{range} T)^\circ = \{0\}$

Suppose $(\operatorname{range} T)^\circ = \{0\}$, then $\dim (\operatorname{range} T)^\circ = 0$, so by 3.106 of Axler, $\dim(\operatorname{range} T) = \dim W$.

Hence, proved.

3.109 $T \in \mathcal{L}(V, W)$

(a) $\dim \operatorname{range} T' = \dim \operatorname{range} T$

(b) $\operatorname{range} T' = (\operatorname{null} T)^\circ$

<u>Proof.</u> (a) $\dim \operatorname{range} T' = \dim W' - \dim \operatorname{null} T'$

$= \dim W - \dim(\operatorname{null} T')$

$= \dim W - \dim(\operatorname{range} T)^\circ$ by 3.107

$= \dim \operatorname{range} T$ by 3.106

(b) suppose $\varphi \in \operatorname{range} T'$ Then $\exists \psi \in W'$ that satisifies

$\varphi = T'(\psi)$ For all $v \in \operatorname{null} T$

$\varphi(v) = (T'(\psi))(v)$

$= (\psi \circ T)(v)$

$= \psi(Tv)$

$= \psi(0) = 0$

So, $\varphi \in (\operatorname{null} T)^\circ$ Therefore, $\operatorname{range} T' \subset (\operatorname{null} T)^\circ$

Show $\dim \operatorname{range} T' = \dim (\operatorname{null} T)^\circ$

$\dim \operatorname{range} T' = \dim \operatorname{range} T = \dim V - \dim \operatorname{null} T$

$= \dim (\operatorname{null} T)^\circ$ by 3.106

So, $\operatorname{range} T' = (\operatorname{null} T)^\circ$

3.110 T injective $\equiv$ T' is surjective.

Proof: T is injective

iff $\operatorname{null} T = \{0\}$

iff $(\operatorname{null} T)^{\circ} = V'$

iff $\operatorname{range} T' = V'$ by 3.109

iff T' is surjective by 3.20

Proof: $(\operatorname{null} T)^{\circ} = V'$

$$\begin{aligned} \dim (\operatorname{null} T)^{\circ} &= \dim V - \dim \operatorname{null} T \\ &= \dim V - \dim \{0\} \\ &= \dim V \\ &= \dim V' \quad \text{by } 3.95 \end{aligned}$$

3.111 A^t is the transpose of A.

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} \qquad A^T = \begin{bmatrix} A_{1,1} & & A_{m,1} \\ \vdots & & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{bmatrix}$$

3.113 $(AC)^t = C^t A^t$

Proof: Suppose $k = 1, \dots, p$ and $j = 1, \dots, m$

$$\begin{aligned} ((AC)^t)_{k,j} &= (AC)_{j,k} \\ &= \sum_{r=1}^{n} A_{j,r} C_{r,k} = \sum_{r=1}^{n} (A^t)_{r,j} (C^t)_{k,r} \\ &= \sum_{r=1}^{n} (C^t)_{k,r} (A^t)_{r,j} = (C^t A^t)_{k,j} \quad \text{qed.} \end{aligned}$$

3.114 Suppose $T \in \mathcal{L}(V,W)$. Then $M(T') = (M(T))^T$

Proof: let $A = M(T)$ and $C = M(T')$ Suppose $1 \le j \le m$ and $1 \le k \le n$

By def. of $M(T')$ $\quad T'(\psi_j) = \sum_{r=1}^{n} C_{r,j} \varphi_r$ $\quad T'(\psi_j) = \psi_j \circ T$.

3.32 $\downarrow$

$Tv_k = \sum_{r=1}^{m} A_{r,k} w_r$

$T'(\psi_j) = \sum_{r=1}^{n} C_{r,j} \varphi_r$

$$\begin{aligned} \text{So, } (\psi_j \circ T)(v_k) &= \sum_{r=1}^{n} C_{r,j} \varphi_r (v_k) \\ &= (C_{1,j} \varphi_1 + \dots + C_{n,j} \varphi_n)(v_k) \\ &= C_{1,j} \underbrace{\cancel{\varphi_1(v_k)}}_{0} + \dots C_{k,j} \underbrace{\cancel{\varphi_k(v_k)}}_{1} + \dots + C_{n,j} \underbrace{\cancel{\varphi_n(v_k)}}_{0} \\ &= C_{k,j} \end{aligned}$$

$$\begin{aligned} \text{And } (\psi_j \circ T)(v_k) &= \psi_j (T v_k) \\ &= \psi_j \left(\sum_{r=1}^{m} A_{r,k} w_r \right) = \sum_{r=1}^{m} A_{r,k} \psi_j (w_r) \\ &= A_{j,k} \end{aligned}$$

So, $C_{k,j} = A_{j,k}$, thus $C = A^t$

That is $M(T') = (M(T))^t$ $\underline{qed}$