3.F Duality

3.92 Def

- a linear functional on V is a linear map $\varphi: V \to \mathbb{F}$. In other words, $\varphi \in \mathcal{L}(V, \mathbb{F})$

3.93 Example

Define $\varphi: \mathbb{R}^3$ by $\mathbb{R}$ by

$$\varphi(x, y, z) = 4x - 5y + 2z$$

Then φ is a linear functional on $\mathbb{R}^3$

For some $c_1, \dots, c_n \in \mathbb{F}$, the map $\varphi: \mathbb{F}^n \to \mathbb{F}$ defined by $\varphi(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$ then φ is a linear functional on $\mathbb{F}^n$

- Define $\varphi: P(\mathbb{R}) \to \mathbb{R}$ by

$$\varphi(p) = \int_0^1 p(x)\,dx$$

Then φ is a linear functional on $P(\mathbb{R})$

3.94 Def

The dual space of V, denote V', is the vector space of all linear functionals on V. In other words,

$$V' = \mathcal{L}(V, \mathbb{F})$$

3.95 $\dim V' = \dim V$

Suppose V is finite-dim. Then V' is also finite-dim. and $\dim V' = \dim V$

Proof: $\dim V' = \dim \mathcal{L}(V, \mathbb{F})$

$= (\dim V)(\dim \mathbb{F})$

$= (\dim V) \cdot 1$

$= \dim V$ ∎

3.96 Def

If $v_1, \dots, v_n$ is a basis of V, then the dual basis of $v_1, \dots, v_n$ is the list $\varphi_1, \dots, \varphi_n$ of elements of V', where each φ_j for any $j = 1, \dots, n$ is a linear functional on V that satisfies

$$\varphi_j(v_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

3.97 Example

What is the dual basis of the standard bases $e_1, \dots, e_n$ of $\mathbb{F}^n$?

$$e_1 = (1, 0, \dots, 0)$$
$$e_2 = (0, 1, 0, \dots, 0)$$
$$e_3 = (0, 0, 1, 0, \dots, 0)$$
$$\vdots$$
$$e_n = (0, \dots, 0, 1)$$

Solution: $\forall\, j = 1, \dots, n$ write

$$\varphi_j(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$$

Then $1 = \varphi_1(v_1) = \varphi_1(e_1) = \varphi_1(1, 0, \dots, 0) = c_1(1) + c_2(0) + \dots + c_n(0) = c_1$

$0 = \varphi_1(v_2) = \varphi_1(e_2) = \varphi_1(0, 1, 0, \dots, 0) = c_1(0) + c_2(1) + c_3(0) + \dots + c_n(0) = c_2$

$\vdots$

$0 = \varphi_1(v_n) = \varphi_1(e_n) = \varphi_1(0, \dots, 0, 1) = c_1(0) + \dots + c_{n-1}(0) + c_n(1) = c_n$

So $\varphi_1(x_1, \dots, x_n) = (1)x_1 + (0)x_2 + \dots + (0)x_n$

Similarly

$$\varphi_1(x_1, \dots, x_n) = x_1$$
$$\varphi_2(x_1, \dots, x_n) = x_2$$
$$\varphi_3(x_1, \dots, x_n) = x_3$$
$$\vdots$$
$$\varphi_n(x_1, \dots, x_n) = x_n$$

So $\varphi_1, \dots, \varphi_n$ as defined above is the dual basis of $v_1, \dots, v_n$

3.98 Dual Basis is a basis of the dual space

Suppose V is finite-dim.

Then the dual basis of a basis of V is a basis of V'.

Proof: let $v_1, \dots, v_n$ be a basis of V, and let $\varphi_1, \dots, \varphi_n$ be a dual basis of $v_1, \dots, v_n$.

We will show $\varphi_1, \dots, \varphi_n$ is LI.

Suppose $a_1, \dots, a_n \in \mathbb{F}$ satisfy

$a_1\varphi_1 + \dots + a_n\varphi_n = 0$

Then, for any $j = 1, \dots, n$ we have

$(a_1\varphi_1 + \dots + a_n\varphi_n)(v_j) = a_1\varphi_1(v_j) + \dots + a_n\varphi_n(v_j)$

$= a_1\varphi_1(v_j) + \dots + a_j\varphi_j(v_j) + \dots + a_n\varphi_n(v_j)$

$= a_1 \cdot 0 + \dots + a_j \cdot 1 + \dots + a_n \cdot 0 = a_j$

So $a_j = (a_1\varphi_1 + \dots + a_n\varphi_n)v_j = 0(v_j) = 0$

In other words,

$a_1 = 0, \dots, a_n = 0$

So $\varphi_1, \dots, \varphi_n$ is LI.

By 3.95, $\dim V' = \dim V$

By 2.39 $\varphi_1, \dots, \varphi_n$ is a basis of V' ∎

3.99 Def

If $T \in \mathcal{L}(V, W)$, then the dual map of T is the linear map $T' \in \mathcal{L}(W', V')$

defined by

$T'(\varphi) = \varphi \circ T$

Show: $T' \in \mathcal{L}(W', V')$

let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in W'$ be arbitrary.

- Additivity: $T'(\varphi + \psi) = (\varphi + \psi) \circ T$

  $= \varphi \circ T + \psi \circ T$

  $= T'(\varphi) + T'(\psi)$

- Homogeneity: $T'(\lambda\varphi) = (\lambda\varphi) \circ T = \lambda(\varphi \circ T) = \lambda T'(\varphi)$.

3.100 Example)

Define $D: P(\mathbb{R}) \to P(\mathbb{R})$ by $D(p) = p'$

· Define $\varphi \in \mathcal{L}(P(\mathbb{R}), \mathbb{F})$ by $\varphi(p) = p(3)$.

Then $D'(\varphi)$ is the linear dim. on $P(\mathbb{R})$ given by

$$D'(\varphi)(p) = (\varphi \circ D)(p)$$
$$= \varphi(Dp) = \varphi(p') = \int_0^1 p'(x)\,dx = p(1) - p(0)$$

Define $\varphi \in \mathcal{L}(P(\mathbb{R}), \mathbb{F})$ by

$$\varphi(p) = \int_0^1 p(x)\,dx$$

Then $D'(\varphi)$ is the linear functional on $P(\mathbb{R})$ given by

$$(D'(\varphi))(p) = (\varphi \circ D)(p)$$
$$= \varphi(Dp) = \varphi(p') = \int_0^1 p'(x)\,dx = p(1) = p(0).$$

3.101 Algebraic Prop. Dual Maps

a) $(S+T)' = S' + T'$ $\forall$ $S, T \in \mathcal{L}(V, W)$

b) $(\lambda T)' = \lambda T'$ $\forall$ $\lambda \in \mathbb{F}$ and $\forall$ $T \in \mathcal{L}(V, W)$

c) $(ST)' = T'S'$ $\forall T \in \mathcal{L}(U, V)$ and $\forall$ $S \in \mathcal{L}(V, W)$

Proof: a) $\forall$ $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$(S+T)'(\varphi) = \varphi \circ (S+T) = \varphi \circ S + \varphi \circ T = S'(\varphi) + T'(\varphi)$$

b) $\forall$ $\varphi \in \mathcal{L}(V, \mathbb{F})$,

$$(\lambda T)'(\varphi) = \varphi \circ (\lambda T) = \lambda \varphi \circ T = \lambda T'(\varphi)$$

c) $\forall$ $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$(ST)'(\varphi) = \varphi \circ (ST)$$
$$= (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'(\varphi))$$
$$= (T'S')(\varphi)$$

So $(ST)'(\varphi) = T'S'$

3.102 Def)

If U is a subspace of V, then the annihilator of U

denoted U^0, is defined by

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \ \forall u \in U\}$$

3.103 Ex)

let V be the subspace of $P(\mathbb{R})$
consisting of polynomials of x^2,
such as $x^2p(x)$
If $\varphi \in \mathcal{L}(P(\mathbb{R}), \mathbb{F})$ is defined by $\varphi(p) = p'(0)$,
then $\varphi \in V^0$

$x^2p(x) \in V$, if $p \in P(\mathbb{R}), \mathbb{F})$ is defined by $\varphi(a) = p'(0)$,
the $\varphi \in V^0$

3.104 Ex)

let $e_1, e_2, \ldots, e_5$ denote the standard basis of $\mathbb{R}^5$,
and let $\varphi_1, \varphi_2, \ldots, \varphi_5$ denote the dual basis of $(\mathbb{R}^5)'$.
Suppose
$$V = \operatorname{span}(e_1, e_2) = \{(x_1, x_2, 0, 0, 0) \in \mathbb{R}^5 : x_1, x_2 \in \mathbb{R}\}$$
Show $V^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.
Solution. Recall from Example 3.97 that
$$\varphi_j(x_1, x_2, x_3, \ldots, x_5) = x_j$$
for any $j = 1, 2, \ldots, 5$.
Suppose we have $\varphi \in \operatorname{span}(\varphi_3, \ldots, \varphi_5)$
Then $\varphi = c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$
for some $c_3, c_4, c_5 \in \mathbb{R}$. $\forall (x_1, x_2, 0, 0, 0) \in V$.
we have $\varphi(x_1, x_2, 0, 0, 0) = (c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(x_1, x_2, 0, 0, 0)$
$$= c_3\varphi_3(x_1, x_2, 0, 0, 0) + c_4\varphi_4(x_1, x_2, 0, 0, 0)$$
$$+ c_5\varphi_5(x_1, x_2, 0, 0, 0)$$
$$= c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 = 0$$
so $\varphi \in V^0$, and so $\operatorname{span}(\varphi_3, \varphi_4, \varphi_5) \subset V^0$
Suppose $\varphi \in V^0$. Since $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ is the dual basis
of $(\mathbb{R}^5)'$, we can write uniquely as $\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + \ldots + c_5\varphi_5$
for some $c_1, c_2, \ldots, c_5 \in \mathbb{R}$
Since $e_1 \in V$, we have
$$0 = \varphi(e_1)$$
$$= (c_1\varphi_1 + c_2\varphi_2 + \ldots + c_5\varphi_5)(e_1)$$

$= c_1\varphi_1(e_1) + (c_2\varphi_2 + \dots + c_5\varphi_5)(e_1)$

$= c_1\varphi_1(e_1) + \dots + c_5(e_1)$

$= c_1 \cdot 1 + c_2 \cdot 0 + \dots + c_5 \cdot 0 = c_1$

Since $e_2 \in U$,

$0 = \varphi(e_2)$

$= (c_1\varphi_1 + c_2\varphi_2 + \dots c_5\varphi_5)(e_2)$

$= c_1\varphi_1(e_2) + \dots + c_5\varphi_5(e_5)$

$= c_1 \cdot 0 + c_2 \cdot 1 + \dots + c_5 \cdot 0 = c_2$

Therefore,

$\varphi = c_1\varphi_1 + \dots + c_5\varphi_5$

$= 0\varphi_1 + \dots 0\varphi_2 + \dots + c_5\varphi_5$

$= c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

So $U^0 \subset \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

So, $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

3.105 The annihilator of a subspace

Suppose U is a subspace of V.

The U^0 is a subspace of V'.

$(V' = \mathcal{L}(V, \mathbb{F}))$

Proof:

- Additive identity: Since $0(u) = 0 \ \forall u \in U$, we have $0 \in U^0$.
- Closed under addition: Suppose $\varphi, \psi \in U^0 \ \forall u \in U$. we have

  $(\varphi + \psi)(u) = \varphi(u) + \psi(u) = 0 + 0 = 0$

  So $\varphi + \psi \in U^0$
- Closed under scalar multiplication: suppose $\lambda \in \mathbb{F}$ and $\varphi \in U^0 \ \forall u \in U$,

  We have $(\lambda\varphi)(u) = \lambda\varphi(u) = \lambda \cdot 0 = 0$

  So $\lambda\varphi \in U^0$ ∎

3.111 Def)

The transpose of a matrix A, denoted A^t, is the matrix obtained from A by interchanging the rows and columns

More specifically, if

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix},$$

then

$$A^t = \begin{bmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{bmatrix}$$

3.112 Ex)

If $A = \begin{bmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{bmatrix}$, then $A^t = \begin{bmatrix} 5 & 3 & -4 \\ -7 & 8 & 2 \end{bmatrix}$

3.113 The transpose of the product of matrices

let A be an $m \times n$ matrix and B be a $n \times T$ matrix. then..

$$(AC)^t = C^t A^t$$

Proof: Suppose we have

$k = 1, \dots, p$ and $i, \dots, m$

$$((AC)^t)_{k,i} = (AC)_{i,k}$$
$$= \sum_{r=1}^{n} A_{i,r} \cdot C_{r,k}$$

$$= \sum_{r=1}^{n} (A^t)_{r,i} \cdot (C^t)_{k,r}$$
$$= \sum_{r=1}^{n} (C^t)_{k,r} (A^t)_{r,i}$$
$$= (C^t A^t)_{k,i}$$

Therefore, we have $(AC)^t = C^t A^t$ $\boxed{\text{Q.E.D.}}$

3.114 The matrix of T' is the transpose of the matrix of T.

Suppose $T \in \mathcal{L}(V, W)$ then $M(T') = (M(T))^t$

Proof: let $A = M(T)$ and $C = M(T')$

Suppose we have $j = 1, \dots, m$ and $k = 1, \dots, n$.

By def 3.32, we have

$$Tv_k = \sum_{r=1}^{m} A_{r,k} w_r$$

$$\text{and} \quad T'(\varphi_j) = \sum_{r=1}^{n} C_{r,j} \varphi_r$$

So we have

$$\begin{aligned}
(\varphi_j \circ T)(v_k) &= (T'(\varphi_j))(v_k) \\
&= \Big(\sum_{r=1}^{n} C_{r,j} \varphi_r\Big)(v_k) \\
&= (C_{1,j}\varphi_1 + \dots + C_{n,j}\varphi_n)(v_k) \\
&= C_{1,j}\varphi_1(v_k) + \dots + C_{n,j}\varphi_n(v_k) \\
&= C_{1,j}\varphi_1(v_k) + \dots + C_{k,j}\varphi_k(v_k) + \dots + C_{n,j}\varphi_n(v_k) \\
&= C_{1,j} \cdot 0 + \dots + C_{k,j} \cdot 1 + \dots + C_{n,j} \cdot 0 \\
&= C_{k,j}
\end{aligned}$$

and

$$\begin{aligned}
(\varphi_j \circ T)(v_k) &= \varphi_j(Tv_k) \\
&= \varphi_j\Big(\sum_{r=1}^{n} A_{r,k} w_r\Big) \\
&= \varphi_j(A_{1,k}w_1 + \dots + A_{n,k}w_n) \\
&= \varphi_j(A_{1,k}w_1) + \dots + \varphi_j(A_{n,k}w_n) \\
&= A_{1,k}\varphi_j(w_1) + \dots + A_{n,k}\varphi_j(w_n) \\
&= A_{1,k}\varphi_j(w_1) + \dots + A_{j,k}\varphi_j(w_j) + \dots + A_{n,k}\varphi(w_n) \\
&= A_{1,k} \cdot 0 + \dots + A_{j,k} \cdot 1 + \dots + A_{n,k} \cdot 0 = A_{j,k}
\end{aligned}$$

Therefore, we conclude

$C_{k,j} = A_{j,k}$

So $C = A^t$,

and so $M(T') = C$

$= A^t = (M(T))^t$ ∎

Back to annihilator stuff

(3.106 - 3.110)

3.106 Dim of the annihilator

Suppose V is a fin-dim vector space and U is a subspace of V. Then

$$\dim U + \dim U^0 = \dim V.$$

Proof: let $i \in \mathcal{L}(U, V)$ be the inclusion map defined by $i(u) = u$ $\forall u \in U$

First, we will show that i' is linear

let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in V'$.

- Additivity: $\forall u \in U$, we have

$$\begin{aligned}(i'(\varphi + \psi))(u) &= ((\varphi + \psi) \circ i)(u) \\ &= (\varphi + \psi)(i(u)) \\ &= (\varphi + \psi)(u) \\ &= \varphi(u) + \psi(u) \\ &= \varphi(i(u)) + \psi(i(u)) \\ &= (\varphi \circ i)(u) + (\psi \circ i)(u) \\ &= (i'(\varphi))(u) + (i'(\psi))(u) \\ &= (i'(\varphi) + i'(\psi))(u)\end{aligned}$$

Therefore,

$$i'(\varphi + \psi) = i'(\varphi) + i'(\psi)$$

- Homogeneity: $\forall u \in U$, we have

$$\begin{aligned}(i'(\lambda \varphi))(u) &= ((\lambda \varphi) \circ i)(u) \\ &= (\lambda \varphi)(i(u)) \\ &= (\lambda \varphi)(u) \\ &= \lambda \varphi(u) \\ &= \lambda \varphi(i(u)) \\ &= \lambda ((\varphi \circ i)(u)) \\ &(\lambda (i'(\varphi)))(u)\end{aligned}$$

Therefore,

$$i'(\lambda \varphi) = \lambda i'(\varphi)$$

Next, we will show null $i' = V^0$

We have

$$\begin{aligned}
\text{null}\, i' &= \{\varphi \in V' : i'(\varphi) = 0\} \\
&= \{\varphi \in V' : \varphi \circ i = 0\} \\
&= \{\varphi \in V' : (\varphi \circ i)(u) = 0 \ \forall u \in U\} \\
&= \{\varphi \in V' : \varphi(i(u)) = 0 \ \forall u \in U\} \\
&= \{\varphi \in V' : \varphi(u) = 0 \ \forall u \in U\} \\
&= U^0.
\end{aligned}$$

By the fundamental theorem of linear maps (3.22)

$$\begin{aligned}
\dim V = \dim V' \quad &\text{by } 3.95 \\
= \dim \text{range}\, i' + \dim \text{null}\, i' \quad &(3.22) \\
= \dim \text{range}\, i' + \dim U^0 \\
= \dim U + \dim U^0
\end{aligned}$$

once we show range $i' = U'$.

Show range $i' = U'$

Suppose we have $\varphi \in U'$. By 3.A.11, we can extend to a linear functional ψ on V. And by definition of i' we have $i'(\psi) = \varphi$. So $\varphi \in$ range i', and so we have $U' \subset$ range i'. But 3.19 says range i' is a subspace of U'.

So range $i' = U'$, as desired $\boxtimes$

3.107 The nullspace of T'

Suppose V and W are finite-dim vector spaces and $T \in \mathcal{L}(V, W)$ then:

a) null $T' = (\text{range}\ T)^0$

b) $\dim \text{null}\, T' = \dim \text{null}\, T + \dim W - \dim V$.

Proof: a) Suppose $\varphi \in$ null T'. Then we have $T'(\varphi) = 0$. So $\forall v \in V$, we have

$$\begin{aligned}
0 = 0(v) &= (T'(\varphi))(v) \\
&= (\varphi \circ T)(v) = \varphi_0(Tv)
\end{aligned}$$

So $\varphi \in (\text{range}\, T)^0$, and so we have null $T' \subset (\text{range}\, T)^0$

Suppose $\varphi \in (\text{range } T)^0$ Then $\varphi(Tv)=0\ \forall v \in V$

so, $(T'(\varphi))(v) = (\varphi \circ T)(v)$

$= \varphi(0) = 0$

so $\varphi \in \text{null } T'$ So $(\text{range } T)^0 \subset \text{null } T'$

So, $(\text{range } T)^0 = \text{null } T'$

b). we have

$\dim \text{null } T' = \dim (\text{range } T)^0$ by part (a)

$= \dim W - \dim \text{range } T$ by 3.106

$= \dim W - (\dim V - \dim \text{null}(T))$ by 3.22

$= \dim \text{null } T + \dim W - \dim V$ $\blacksquare$

3.108 T surjective equivalent to T' injective

Suppose V and W are finite-dim. vector spaces and $T \in \mathcal{L}(V,W)$ then T is surjective IFF T' is injective.

Proof: $\Leftrightarrow$ = "IFF"

$T \in \mathcal{L}(V,W)$ is surjective $\Leftrightarrow$ range $T = W$

$\Leftrightarrow (\text{range } T)^0 = \{0\}$

$\Leftrightarrow \text{null } T' = \{0\}$ by 3.102

$\Leftrightarrow T'$ is injective.

Prove $(\text{range } T)^0 = \{0\}$

We have

$\dim (\text{range } T)^0 = \dim W - \dim \text{range } T$ by 3.106

$= \dim W - \dim W = 0$

$= \dim \{0\}$,

IFF $(\text{range } T)^0 = \{0\}$ by 2.C.1 Axler. $(\Leftrightarrow)$

3.109 the range T'

Suppose V and W are finite-dim vector spaces and $T \in \mathcal{L}(V,W)$ then

a) $\dim \text{range } T' = \dim \text{range } T$,

b) $\text{range } T' = (\text{null } T)^0$.

Proof:

a): we have

$$\dim \operatorname{range} T' = \dim W' - \dim \operatorname{null} T' \quad (3.22)$$
$$= \dim W - \dim \operatorname{null} T' \quad (3.95)$$
$$= \dim W - \dim (\operatorname{range} T)^0 \quad (3.107\,a)$$
$$= \dim \operatorname{range} T. \quad (3.106)$$

b): Suppose $\varphi \in \operatorname{range} T'$

Then $\exists$ $\psi \in W'$ that satisfies $\varphi \in T'(\psi)$. $\forall$ $v \in \operatorname{null} T$, we have

$$\varphi(v) = (T'(\psi))(v)$$
$$= (\psi \circ T)(v) \quad \text{since } v \in \operatorname{null} T$$
$$= (\psi(Tv)) = \varphi(0) = 0$$

so $\varphi \in (\operatorname{null} T)^0$. Therefore, $\operatorname{range} T' \subset (\operatorname{null} T)^0$

Show $\dim \operatorname{range} T' = \dim (\operatorname{null} T)^0$

We have

$$\dim \operatorname{range} T' = \dim \operatorname{range} T \text{ by } 3.109(a)$$
$$= \dim V - \dim \operatorname{null} T \text{ by } (3.22)$$
$$= \dim (\operatorname{null} T)^0$$

3.110 T injective is equivalent to T' surjective

Suppose V and W are finite-dim. vectors spaces and $T \in \mathcal{L}(V, W)$ then.

T is injective IFF T' is surjective.

Proof: $T \in \mathcal{L}(V, W)$ is injective $\Leftrightarrow \operatorname{null} T = \{0\}$ by 3.16

$\Leftrightarrow (\operatorname{null} T)^0 = V'$

$\Leftrightarrow \operatorname{range} T' = V'$ by 3.109

$\Leftrightarrow T'$ is surjective.

prove $(\operatorname{null} T)^0 = V'$

We have

$$\dim (\operatorname{null} T)^0 = \dim V - \dim \operatorname{null} T \text{ by } 3.106$$
$$= \dim V - \dim \{0\}$$
$$= \dim V - 0$$
$$= \dim V, = \dim V' \text{ by } 3.95,$$

IFF $(\operatorname{null} T)^0 = V'$, by exercise 2.C.1 Axler.