July. 23

3.90 Definition

Suppose $T \in \mathcal{L}(V, W)$. Define $\tilde{T}: V/(\text{null}\, T) \to W$ by

$$\tilde{T}(v + \text{null}\, T) = Tv$$

Show that $\tilde{T}$ makes sense ($\tilde{T}$ is well-defined)

Suppose $u, v \in V$ satisfy

$$u + \text{null}\, T = v + \text{null}\, T$$

By 3.85, we have

$$u - v \in \text{null}\, T$$

This means

$$T(u-v) = 0$$

In fact, we have

$$Tu - Tv = T(u-v) = 0,$$

$\therefore$ $Tu = Tv$.

$$\begin{aligned} \therefore \tilde{T}(u + \text{null}\, T) &= Tu \\ &= Tv \\ &= \tilde{T}(v + \text{null}\, T) \end{aligned}$$

and so $\tilde{T}$ is well-defined.

3.91 Null space and range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V, W)$ Then:

a). $\tilde{T}$ $V/(\text{null}\, T) \to W$ is a linear map; $\tilde{T} \in \mathcal{L}(V/(\text{null}\, T), W)$

b). $\tilde{T}$ is injective.

c). range $\tilde{T}$ = range T

d). $V/(\text{null}\, T)$ is isomorphic to range T.

Proof: (a) Let $u, v \in V$ and $\lambda \in \mathbb{F}$.

• additivity: $\tilde{T}((u+\operatorname{null}T)+(v+\operatorname{null}T))$

$= \tilde{T}((u+v)+\operatorname{null}T)$

$= T(u+v)$

$= Tu + Tv$

$= \tilde{T}(u+\operatorname{null}T) + \tilde{T}(v+\operatorname{null}T)$

• homogeneity: $\tilde{T}(\lambda(v+\operatorname{null}T))$

$= \tilde{T}((\lambda v)+\operatorname{null}T)$

$= T(\lambda v)$

$= \lambda Tv$

$= \lambda \tilde{T}(v+\operatorname{null}T)$

$\therefore \tilde{T}$ is linear.

b). Suppose $v \in V$ and $\tilde{T}(v+\operatorname{null}T) = 0$

Then we have

$Tv = \tilde{T}(v+\operatorname{null}T)$

$= 0$

So $v - 0 = v \in \operatorname{null}T$

By 3.85

$v+\operatorname{null}T = 0+\operatorname{null}T$ .... additive identity of $V/\operatorname{null}T$

$\therefore \operatorname{null}\tilde{T} \subset \{0+\operatorname{null}T\}$

$\tilde{T}(0+\operatorname{null}T) = 0$ since $\tilde{T}$ is linear

$\therefore 0+\operatorname{null}T \subset \operatorname{null}\tilde{T}$, or $\{0+\operatorname{null}T\} = \operatorname{null}\tilde{T}$. By 3.16, $\tilde{T}$ is injective.

C). For all $v \in V$, $\tilde{T}(v+\operatorname{null}T) = Tv$

Suppose $w \in \operatorname{range} T$. Then $w = Tv$

for some $v \in V$. In fact,

$w = Tv$

$= \tilde{T}(v + \operatorname{null} T) \in \operatorname{range} \tilde{T}$

So $\operatorname{range} T \subset \operatorname{range} \tilde{T}$

Suppose $x \in \operatorname{rang} \tilde{T}$. Then $x = \tilde{T}(v + \operatorname{null} T)$ for some $v \in V$.

In fact,

$x = \tilde{T}(v + \operatorname{null} T)$

$= Tv \in \operatorname{range} T$

So $\operatorname{range} \tilde{T} \subset \operatorname{range} T$

$\therefore \operatorname{range} \tilde{T} = \operatorname{range} T$

d). By part c), $\operatorname{rang} \tilde{T} = \operatorname{range} \tilde{T}$

If we think of $\tilde{T}$ as a map into $\operatorname{range} \tilde{T}$.

$\tilde{T} : V/(\operatorname{null} T) \to \operatorname{range} T$ is surjective.

So $\tilde{T}$ is also surjective.

By part (b), $\tilde{T}$ is also injective.

$\therefore$ By 3.56, $\tilde{T}$ is invertible.

By part a),. $\tilde{T}$ is linear.

$\therefore \tilde{T} : V/(\operatorname{null} T) \to \operatorname{range} T$ is an isomorphism.

3F. Duality

3.92 Definition

A linear functional on V is a linear map $\varphi : V \to \mathbb{F}$.

In other words, $\varphi \in \mathcal{L}(V, \mathbb{F})$

3.93 Example

- Define $\varphi : \mathbb{R}^3 \to \mathbb{R}$ by

$\varphi(x,y,z) = 4x - 5y + 2z$.

Then φ is a linear functional on $\mathbb{R}^3$.

- For some $c_1, \dots, c_n \in \mathbb{F}$, the map $\varphi: \mathbb{F}^n \to \mathbb{F}$ defined by
$\varphi(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$
then φ is a linear functional on $\mathbb{F}^n$.
- Define $\varphi: P(\mathbb{R}) \to \mathbb{R}$ by
$$\varphi(p) = \int_0^1 p(x)\,dx$$
Then φ is a linear functional on $P(\mathbb{R})$

3.94 Definition

Then dual space of V, denoted V', is the vector space of all linear functionals on V. In other words,
$V' = \mathcal{L}(V, \mathbb{F})$.

3.95 $\dim V' = \dim V$

Suppose V is finite-dimensional. Then V' is also finite-dimensional and $\dim V' = \dim V$

Proof:

$$\begin{aligned}\dim V' &= \dim \mathcal{L}(V, \mathbb{F}) \\ &= (\dim V)(\dim \mathbb{F}) \quad \text{by } 3.61 \\ &= (\dim V) \cdot 1 \\ &= \dim V\end{aligned}$$

3.96 Definition

If $v_1, \dots, v_n$ is a basis of V, then the dual basis of $v_1, \dots, v_n$

is the list $\varphi_1,\dots,\varphi_n$ of elements of V', where each φ_j for any $j=1,\dots,n$ is a linear functional on V that satisfies

$$\varphi_j(v_k)=\begin{cases}1 & k=j\\ 0 & k\neq j\end{cases}$$

3.97 Example

What is the dual basis of the standard basis $e_1,\dots,e_n$ of $\mathbb{F}^n$?

$e_1=(1,0,\dots 0)$

$\vdots$

$e_n=(0,\dots,0,1)$

solution:

For all $j=1,\dots,n$, write

$\varphi_j(x_1,\dots,x_n)=c_1x_1+\dots+c_nx_n$

Then

$1=\varphi_1(v_1)=\varphi_1(e_1)=\varphi_1(1,0,\dots 0)=c_1(1)+c_2(0)+\dots+c_n(0)=c_1$

$0=\varphi_1(v_2)=\varphi_1(e_2)=\varphi(0,1,\dots 0)=c_1(0)+c_2(1)+\dots+c_n(0)=c_2$

$\vdots$

$0=\varphi_1(v_n)=\varphi(e_n)=\varphi_1(0,\dots,0,1)=c_1(0)+\dots+c_{n-1}(0)+c_n(1)=c_n$

So

$\varphi_2(x_1,\dots,x_n)=(1)x_1+(0)x_2+\dots+(0)x_n$

$\quad =x_1$

Similarly

$\varphi_1(x_1,\dots,x_n)=x_1$

$\varphi_2(x_1,\dots,x_n)=x_2$

$\vdots$

$\varphi_n(x_1,\ldots,x_n)=x_n$

So $\varphi_1,\ldots,\varphi_n$ as defined above is the dual basis of $v_1,\ldots,v_n$

Axler Define φ_j to be the linear functional on $\mathbb{F}^n$ that selects the j^{th} coordinate of a vector in $\mathbb{F}^n$.

In other words,

$\varphi_j(x_1,\ldots,x_n)=x_j$ for all $(x_1,\ldots,x_n)\in\mathbb{F}^n$.

3.98 Dual basis is a basis of the dual space.

Suppose V is finite-dimensional.

Then the dual basis of a basis of V is a basis of V'.

Proof:

Let $v_1,\ldots,v_n$ be a basis of V, and let $\varphi_1,\ldots,\varphi_n$ be a dual basis of $v_1,\ldots,v_n$.

We will show that $\varphi_1,\ldots,\varphi_n$ is linearly independent.

Suppose $a_1,\ldots,a_n\in\mathbb{F}$ satisfy

$$a_1\varphi_1+\ldots+a_n\varphi_n=0$$

Then, for any $j=1,\ldots,n$ we have

$$(a_1\varphi_1+\ldots+a_n\varphi_n)(v_j)=a_1\varphi_1(v_j)+\ldots+a_n\varphi_n(v_j)$$
$$=a_1\varphi_1(v_j)+\ldots+a_j\varphi_j(v_j)+\ldots+a_n\varphi_n(v_n)$$
$$=a_1\cdot 0+\ldots+a_j\cdot 1+\ldots+a_n\cdot 0$$
$$=a_j$$

So we have

$$a_j=(a_1\varphi_1+\ldots+a_n\varphi_n)(v_j)$$
$$=0\cdot(v_j)$$

$= 0$

In other words,

$a_1 = 0, \ldots, a_n = 0$

So $\varphi_1, \ldots, \varphi_n$ is linearly independent.

By 3.95, $\dim V' = \dim V$

By 2.39. $\varphi_1, \ldots, \varphi_n$ is a basis of V'.

3.99 Definition

If $T \in \mathcal{L}(V, W)$, then the dual map of T is the linear map $T' \in \mathcal{L}(W', V')$ defined by

$T'(\varphi) = \underline{\varphi \circ T}$

input a linear functional

Out put a linear functional

Show: $T' \in \mathcal{L}(W', V')$

Let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in W'$ be arbitrary.

- additivity: $T'(\varphi + \psi) = (\varphi + \psi) \circ T$
$= \varphi \circ T + \psi \circ T$
$= T'(\varphi) + T'(\psi)$
- homogenerty: $T'(\lambda \varphi) = (\lambda \varphi) \circ T$
$= \lambda (\varphi \circ T)$
$= \lambda T'(\varphi)$

3.100 Example

Define $D: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $Dp = p'$

- Define: $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{F})$ by

$$\varphi(p) = p(3)$$

Then $D'(\varphi)$ is the linear functional on $P(\mathbb{R})$ that satisfies

$$\begin{aligned}(D'(\varphi))(p) &= (\varphi \circ D)(p) \\ &= \varphi(Dp) \\ &= \varphi(p') \\ &= p'(3)\end{aligned}$$

- Define $\varphi \in \mathcal{L}(P(\mathbb{R}), \mathbb{F})$ by

$$\varphi(p) = \int_0^1 p(x)\,dx.$$

Then $D'(\varphi)$ is then linear functional on $P(\mathbb{R})$ given by

$$\begin{aligned}(D'(\varphi))(p) &= (\varphi \circ D)(p) \\ &= \varphi(Dp) \\ &= \varphi(p') \\ &= \int_0^1 p'(x)\,dx \\ &= p(1) - p(0).\end{aligned}$$

3.101 Algebraic Properties of Dual Maps

a. $(S+T)' = S' + T'$ for all $S, T \in \mathcal{L}(V, W)$

b. $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$ and for all $T \in \mathcal{L}(V, W)$

c. $(ST)' = T'S'$ for all $T \in \mathcal{L}(U, V)$ and for all $S \in \mathcal{L}(V, W)$

Proof:

a). for all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$\begin{aligned}(S+T)'(\varphi) &= \varphi \circ (S+T) \\ &= \varphi \circ S + \varphi \circ T \\ &= S'(\varphi) + T'(\varphi)\end{aligned}$$

b). For all $\varphi \in \mathcal{L}(V, \mathbb{F})$

$$(\lambda T)'(\varphi) = \varphi \circ (\lambda T)$$
$$= \lambda \varphi \circ T$$
$$= \lambda T'(\varphi)$$

c). For all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$(ST)'(\varphi) = \varphi \circ (ST)$$
$$= (\varphi \circ S) \circ T$$
$$= T'(\varphi \circ S)$$
$$= T'(S'(\varphi))$$
$$= (T'S')(\varphi)$$

So $(ST)'(\varphi) = T'S'$

3.102 Definition

If U is a subspace of V, then the annihilator of U, denoted U^0, is defined by

$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$

3.103 Example

Let U be the subspace of $P(\mathbb{R})$ consisting of all polynomial of x^2, such as $x^2 p(x)$.

If $\varphi \in \mathcal{L}(P(\mathbb{R}), \mathbb{F})$ is defined by

$\varphi(0) = P'(0)$. then $\varphi \in U^0$

$x^2 p(x) \in U$ if $p \in P(\mathbb{R})$

And $\varphi(x^2 p(x)) = (x^2 p(x))' |_{x=0}$

$$= (2x p(x) + x^2 p'(x))|_{x=0}$$

$= 2(0)p(0) + 0^2 p'(0)$

$= 0$

3.104 Example

Let e_1, e_2, e_3, e_4, e_5 denote the standard basis of $\mathbb{R}^5$, and let $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ denote the dual basis of $(\mathbb{R}^5)'$

Suppose

$$U = \text{Span}(e_1, e_2)$$
$$= \{(x_1, x_2, 0, 0, 0) \in \mathbb{R}^5 : x_1, x_2 \in \mathbb{R}\}$$

Show $U^0 = \text{span}(\varphi_3, \varphi_4, \varphi_5)$

Solution:

Recall from Example 3.97 that

$\varphi_j(x_1, x_2, x_3, x_4, x_5) = x_j$

for any $j = 1, 2, 3, 4, 5$

Suppose we have $\varphi \in \text{span}(\varphi_3, \varphi_4, \varphi_5)$

Then $\varphi = c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$

for some $c_3, c_4, c_5 \in \mathbb{R}$. For all $(x_1, x_2, 0, 0, 0) \in U$.

We have $\varphi(x_1, x_2, 0, 0, 0) = (c_3\varphi_3, c_4\varphi_4, c_5\varphi_5)(x_1, x_2, 0, 0, 0)$

$$= c_3\varphi_3(x_1, x_2, 0, 0, 0) + c_4\varphi_4(x_1, x_2, 0, 0, 0) + c_5\varphi_5(x_1, x_2, 0, 0, 0)$$
$$= c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0$$
$$= 0$$

$\therefore \varphi \in U^0$, and so $\text{span}(\varphi_3, \varphi_4, \varphi_5) \subset U^0$.

Suppose $\varphi \in U^0$. Since $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ is the dual basis of $(\mathbb{R}^5)'$, we can write uniquely as

$\varphi = c_1\varphi_1 + \dots + c_5\varphi_5$ for some $c_1, \dots, c_5 \in \mathbb{R}$.

Since $e_1 \in U$, we have

$$
\begin{aligned}
0 &= \varphi(e_1) \\
&= (c_1\varphi_1 + c_2\varphi_2 + \dots + c_5\varphi_5)(e_1) \\
&= c_1\varphi_1(e_1) + c_2\varphi_2(e_1) + c_3\varphi_3(e_1) + c_4\varphi_4(e_1) + c_5\varphi_5(e_1) \\
&= c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
&= c_1
\end{aligned}
$$

Since $e_2 \in U$,

$$
\begin{aligned}
0 &= \varphi(e_2) \\
&= (c_1\varphi_1 + \dots + c_5\varphi_5)(e_2) \\
&= c_1\varphi_1(e_2) + c_2\varphi_2(e_2) + c_3\varphi_3(e_2) + c_4\varphi_4(e_2) + c_5\varphi_5(e_2) \\
&= c_1 \cdot 0 + c_2 \cdot 1 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
&= c_2
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \varphi &= c_1\varphi_1 + \dots + c_5\varphi_5 \\
&= 0\varphi_1 + 0\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \\
&= c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \\
&\in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)
\end{aligned}
$$

So $U^0 \subset \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

$\therefore\ U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

3.105 The annihilator is a subspace

Suppose U is a subspace of V.

Then U^0 is a subspace of V'.

($V' = \mathcal{L}(V, \mathbb{F})$)

Proof:

Additive identity: Since $0(u) = 0$ for all $u \in U$, we have $0 \in U^{0}$.

Closed under addition: Suppose $\varphi, \psi \in U^{0}$. For all $u \in U$,

we have

$$
\begin{aligned}
(\varphi+\psi)(u) &= \varphi(u)+\psi(u) \\
&= 0+0 \\
&= 0
\end{aligned}
$$

So $\varphi+\psi \in U^{0}$.

Closed under scalar multiplication: Suppose $\lambda \in \mathbb{F}$ and $\varphi \in U^{0}$. for all $u \in U$

we have

$$
\begin{aligned}
(\lambda\varphi)(u) &= \lambda\varphi(u) \\
&= \lambda \cdot 0 \\
&= 0
\end{aligned}
$$

So $\lambda\varphi \in U^{0}$.

July. 24.

3.111 Definition

The transpose of a matrix A, denoted A^t, is the matrix obtained from A by interchanging the rows and columns.

More specifically, if

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix}$$

then

$$A^t = \begin{bmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{bmatrix}$$

3.112 Example

If $A = \begin{bmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{bmatrix}$, then $A^t = \begin{bmatrix} 5 & 3 & -4 \\ -7 & 8 & 2 \end{bmatrix}$

3.113 The transpose of the product of matrices

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix

Then

$(AC)^t = C^t A^t$

Proof:

Suppose we have $k=1,\dots,p$ and $j=1,\dots,m$

Then, $((AC)^t)_{k,j} = (AC)_{j,k}$

$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

$$= \sum_{r=1}^{n} (A^t)_{r,j} (C^t)_{k,r}$$

$$= \sum_{r=1}^{n} (C^t)_{k,r} (A^t)_{r,j}$$

$$= (C^t A^t)_{k,j}$$

Therefore, we have

$(AC)^t = C^t A^t$

3.114 The matrix of T' is the transpose of the matrix of T.

Suppose $T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(T') = (\mathcal{M}(T))^t$

Proof:

Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$.

Suppose we have $j=1,\dots,m$ and $k=1,\dots,n$

By definition 3.32, we have.

$Tv_k = \sum_{r=1}^{m} A_{r,k} w_r$ and

$T'(\varphi_j) = \sum_{r=1}^{n} C_{r,j} \varphi_r$

Therefore, we have

$$(\varphi_j \circ T)(v_k) = (T'(\varphi_j))(v_k)$$

$$= \left(\sum_{r=1}^{n} C_{r,j} \varphi_r\right)(v_k)$$

$$= (C_{1,j} \varphi_1 + \dots + C_{n,j} \varphi_n)(v_k)$$

$$
\begin{aligned}
&= C_{1,j}\varphi_1(v_k) + \dots + C_{n,j}\varphi_n(v_k)\\
&= C_{1,j}\varphi_1(v_k) + \dots + C_{k,j}\varphi_k(v_k) + \dots + C_{n,j}\varphi_n(v_k)\\
&= C_{1,j}\cdot 0 + \dots + C_{k,j}\cdot 1 + \dots + C_{n,j}\cdot 0\\
&= C_{k,j}
\end{aligned}
$$

and

$$
\begin{aligned}
(\varphi_j \circ T)(v_k) &= \varphi_j(Tv_k)\\
&= \varphi_j\Big(\sum_{r=1}^{n} A_{r,k} w_r\Big)\\
&= \varphi_j(A_{1,k}w_1 + \dots + A_{n,k}w_n)\\
&= \varphi_j(A_{1,k}w_1) + \dots + \varphi_j(A_{n,k}w_n)\\
&= A_{1,k}\varphi_j(w_1) + \dots + A_{n,k}\varphi_j(w_n)\\
&= A_{1,k}\overset{0}{\varphi_j(w_1)} + \dots + A_{j,k}\overset{1}{\varphi_j(w_j)} + \dots + A_{n,k}\overset{0}{\varphi_j(w_n)}\\
&= A_{1,k}\cdot 0 + \dots + A_{j,k}\cdot 1 + \dots + A_{n,k}\cdot 0\\
&= A_{j,k}
\end{aligned}
$$

Therefore, we conclude,

$C_{k,j} = A_{j,k}$

So $C = A^t$

and so

$$
\begin{aligned}
M(T') &= C\\
&= A^t\\
&= (M(T))^t
\end{aligned}
$$

Annihilator

3.106 Dimension of the annihilator

Suppose V is a finite-dimensional vector space and U is a subspace of V. Then

$\dim U + \dim U^0 = \dim V$.

Proof:

Let $i \in \mathcal{L}(U, V)$ be the inclusion map defined by $i(u) = u$ for all $u \in U$.

First, we will show that i is linear.

Let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in V'$.

- Additivity: For all $u \in U$, we have

$$\begin{aligned}
(i'(\varphi+\psi))(u) &= ((\varphi+\psi)\circ i)(u) \\
&= (\varphi+\psi)(i(u)) \\
&= (\varphi+\psi)(u) \\
&= \varphi(u) + \psi(u) \\
&= \varphi(i(u)) + \psi(i(u)) \\
&= (\varphi\circ i)(u) + (\psi\circ i)(u) \\
&= (i'(\varphi))(u) + (i'(\psi))(u) \\
&= (i'(\varphi) + i'(\psi))(u)
\end{aligned}$$

Therefore,

$i'(\varphi+\psi) = i'(\varphi) + i(\psi)$

- Homogeneity: For all $u \in U$, we have

$$\begin{aligned}
(i'(\lambda\varphi))(u) &= ((\lambda\varphi)\circ i)(u) \\
&= (\lambda\varphi)(i(u)) \\
&= (\lambda\varphi)(u) \\
&= \lambda\varphi(u) \\
&= \lambda\varphi(i(u)) \\
&= \lambda((\varphi\circ i)(u)) \\
&= (\lambda(i'(\varphi)))(u)
\end{aligned}$$

Next, we will show $\operatorname{null} i' = U^0$.

We have $\operatorname{null} i' = \{\varphi \in V' : i'(\varphi) = 0\}$

$= \{\varphi \in V' : \varphi \circ i = 0\}$

$= \{\varphi \in V' : (\varphi \circ i)(u) = 0 \text{ for all } u \in U\}$

$= \{\varphi \in V' : \varphi(i(u)) = 0 \text{ for all } u \in U\}$

$= \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$

$= U^0$

By the fundamental theorem of linear maps, we have

$\dim V = \dim V'$ by 3.95

$= \dim \operatorname{range} i' + \dim \operatorname{null} i'$ by 3.22

$= \dim \operatorname{range} i' + \dim U^0$

$= \dim U + \dim U^0$

once we show $\operatorname{range} i' = U'$.

Suppose we have $\varphi \in U'$. By 3A.11, we can extend to a linear functional ψ on V. And by definition of i' we have $i'(\psi) = \varphi$. So $\varphi \in \operatorname{range} i'$, and so we have $U' \subset \operatorname{range} i'$.

But 3.19 says that $\operatorname{range} i'$ is a subspace of U'.

Therefore, $\operatorname{range} i' = U'$, as desired.

3.107 The null space of T'

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then:

a). $\operatorname{null} T' = (\operatorname{range} T)^0$

b). $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$.

Proof: a). Suppose $\varphi \in \text{null } T'$, Then we have:

$T'(\varphi)=0$. So for all $v\in V$, we have

$$\begin{aligned} 0 &= 0(v) \\ &= (T'(\varphi))(v) \\ &= (\varphi \circ T)(v) \\ &= \varphi(Tv) \end{aligned}$$

So $\varphi \in (\text{range } T)^0$, and so we have $\text{null } T' \subset (\text{range } T)^0$.

Suppose $\varphi \in (\text{range } T)^0$. Then $\varphi(Tv)=0$ for all $v \in V$.

So we have
$$\begin{aligned} (T'(\varphi))(v) &= (\varphi \circ T)(v) \\ &= \varphi(Tv) \\ &= \varphi(0) \\ &= 0 \end{aligned}$$

and so $\varphi \in \text{null } T'$. So $(\text{range } T)^0 \subset \text{null } T'$.

Therefore, $(\text{range } T)^0 = \text{null } T'$.

b). We have

$$\begin{aligned} \dim \text{null } T' &= \dim (\text{range } T)^0 && \text{by part a.} \\ &= \dim W - \dim \text{range } T && \text{by } 3.106 \\ &= \dim W - (\dim V - \dim \text{null}(T)) && \text{by. } 3.22 \\ &= \dim \text{null } T + \dim W - \dim V \end{aligned}$$

3.108 T surjective is equivalent to T' injective.

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V,W)$. Then T is surjective iff T' is injective.

Proof: $\Leftrightarrow$ = iff

$T \in \mathcal{L}(V,W)$ is surjective $\Leftrightarrow$ range $T = W$

$\Leftrightarrow (\text{range } T)^0 = \{0\}$

$\Leftrightarrow \text{null } T' = \{0\}$ by 3.107

$\Leftrightarrow T'$ is injective.

Prove $(\text{range } T)^0 = \{0\}$

We have $\dim(\text{range } T)^0 = \dim W - \dim \text{range } T$ by 3.106

$= \dim W - \dim W$

$= 0$

$= \dim \{0\}$

iff $(\text{range } T)^0 = \{0\}$. by 2.C1 of Axler.

3.109 The range of T'

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$ Then

a). $\dim \text{range } T' = \dim \text{range } T$.

b). $\text{range } T' = (\text{null } T)^0$

Proof:

a). We have

$\dim \text{range } T' = \dim W' - \dim \text{null } T'$ by 3.22

$= \dim W - \dim \text{null } T'$ by 3.95

$= \dim W - \dim (\text{range } T)^0$ by 3.107 a.

$= \dim \text{range } T$ by 3.106

b). Suppose $\varphi \in \text{range } T'$. Then there exists $\psi \in W'$ that satisfies. $\varphi = T'(\psi)$. For all $v \in \text{null } T$, we have.

$\varphi(v) = (T'(\psi))(v)$

$= (\psi \circ T)(v)$

$= \varphi(Tv)$

$= \varphi(0)$ since $v \in \operatorname{null} T$

$= 0$

So $\varphi \in (\operatorname{null} T)^0$. Therefore, $\operatorname{range} T' \subset (\operatorname{null} T)^0$

Show $\dim \operatorname{range} T' = \dim (\operatorname{null} T)^0$

We have

$\dim \operatorname{range} T' = \dim \operatorname{range} T$ by 3.109 a.

$= \dim V - \dim \operatorname{null} T$ by 3.22

$= \dim (\operatorname{null} T)^0$ by 3.106

3.110 T injective is equivalent to T' surjective.

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then T is injective iff T' is surjective.

Proof:

$T \in \mathcal{L}(V, W)$ is injective $\Leftrightarrow \operatorname{null} T = \{0\}$ by 3.16

$\Leftrightarrow (\operatorname{null} T)^0 = V'$

$\Leftrightarrow \operatorname{range} T' = V'$ by 3.109 b.

$\Leftrightarrow T'$ is surjective.

Prove $(\operatorname{null} T)^0 = V'$

We have

$\dim (\operatorname{null} T)^0 = \dim V - \dim \operatorname{null} T$ by 3.106

$= \dim V - \dim \{0\}$

$= \dim V - 0$

$= \dim V$

$= \dim V'$ by 3.95

iff $(\text{null}\, T)^0 = V'$, by 2.C.1 exercise.