therefore ... isomorphism

3.F Duality

3.92 Definition:

A linear functional on V is a linear map $\varphi : V \to \mathbb{F}$. In other words, $\varphi \in \mathcal{L}(V, \mathbb{F})$

3.93 Example

Define $\varphi : \mathbb{R}^3 \to \mathbb{R}$ by

$$\varphi(x,y,z) = 4x - 5y + 2z$$

Then φ is a linear functional on $\mathbb{R}^3$.

- For some $c_1, \ldots, c_n \in \mathbb{F}$, the map $\varphi : \mathbb{F}^n \to \mathbb{F}$ defined by

$$\varphi(x_1, \ldots, x_n) = c_1 x_1 + \ldots + c_n x_n,$$

then φ is a linear functional on $\mathbb{F}^n$.

- Define $\varphi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ by

$$\varphi(p) = \int_0^1 p(x)\,dx$$

Then φ is a linear functional on $\mathcal{P}(\mathbb{R})$

3.94 Definition:

The dual space of V, denoted V', is the vector space of all linear functionals on V. In other words,

$$V' = \mathcal{L}(V, \mathbb{F})$$

3.95 $\dim V' = \dim V$

Suppose V is finite-dimensional. Then V' is also finite-dimensional and

$$\dim V' = \dim V$$

Proof: $\dim V' = \dim \mathcal{L}(V, \mathbb{F})$

$= (\dim V)(\dim \mathbb{F})$ by 3.61 of Axler

$= (\dim V) \cdot 1$

$= \dim V$

$\blacksquare$

3.96 Definition:

If $v_1, \dots, v_n$ is a basis of V, then the dual basis of $v_1, \dots, v_n$ is the list $\varphi_1, \dots, \varphi_n$ of elements of V', where each φ_j for any $j = 1, \dots, n$ is a linear functional on V that satisfies

$$\varphi_j(v_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

3.97 Example ☆

What is the dual basis of the standard basis $e_1, \dots, e_n$ of $\mathbb{F}^n$?

$$e_1 = (1, 0, \dots, 0)$$

$$\vdots$$

$$e_n = (0, \dots, 0, 1)$$

Solution: For all $j = 1, \dots, n$ write
$\varphi_j(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$

Then $1 = \varphi_1(v_1) = \varphi_1(e_1) = \varphi_1(1, 0, \dots, 0) = c_1(1) + c_2(0) + \dots + c_n(0) = c_1$

$0 = \varphi_1(v_2) = \varphi_1(e_2) = \varphi_1(0, 1, 0, \dots, 0) = c_1(0) + c_2(1) + \dots + c_n(0) = c_2$

$\vdots$

$0 = \varphi_1(v_n) = \varphi_1(e_n) = \varphi_3(0, \dots, 0, 1) = c_1(0) + \dots + c_{n-1}(0) + c_n(1) = c_n$

So $\varphi_1(x_1, \dots, x_n) = \overset{c_1}{(1)} x_1 + \overset{c_2}{(0)} x_2 + \dots + \overset{c_n}{(0)} x_n = x_1$

$$\boxed{\begin{aligned} c_1 &= 1 \\ c_2 &= 0 \\ &\vdots \\ c_n &= 0 \end{aligned}}$$

Similarly,
$\varphi_1(x_1, \dots, x_n) = x_1$
$\varphi_2(x_1, \dots, x_n) = x_2$
$\varphi_3(x_1, \dots, x_n) = x_3$
$\vdots$
$\varphi_n(x_1, \dots, x_n) = x_n$

So $\varphi_1, \dots, \varphi_n$ as defined above is the dual basis of $v_1, \dots, v_n$

Axler
Define φ_j to be the linear functional on $\mathbb{F}^n$ that selects the jth coordinate of a vector in $\mathbb{F}^n$. In other words, $\varphi_j(x_1, \dots, x_n) = x_j$ for all $(x_1, \dots, x_n) \in \mathbb{F}^n$

3.98 Dual basis is a basis of the dual space

Suppose V is finite-dimensional.
Then the dual basis of a basis of V is a basis of V'.

Proof: Let $v_1, \dots, v_n$ be a basis of V,
and let $\varphi_1, \dots, \varphi_n$ be a dual basis of $v_1, \dots, v_n$.
We will show that $\varphi_1, \dots, \varphi_n$ is linearly independent.
Suppose $a_1, \dots, a_n \in \mathbb{F}$ satisfy

$$a_1 \varphi_1 + \dots + a_n \varphi_n = 0$$

Then, for any $j = 1, \dots, n$ we have

$$\begin{aligned}(a_1 \varphi_1 + \dots + a_n \varphi_n)(v_j) &= a_1 \varphi_1(v_j) + \dots + a_n \varphi_n(v_j) \\ &= a_1 \varphi_1(v_j) + \dots + a_j \varphi_j(v_j) + \dots + a_n \varphi_n(v_j) \\ &= a_1 \cdot 0 + \dots + a_j \cdot 1 + \dots + a_n \cdot 0 \\ &= a_j\end{aligned}$$

So we have

$$\begin{aligned}a_j &= (a_1 \varphi_1 + \dots + a_n \varphi_n)(v_j) \\ &= 0(v_j) \\ &= 0\end{aligned}$$

In other words,

$$a_1 = 0, \dots, a_n = 0$$

So $\varphi_1, \dots, \varphi_n$ is linearly independent.
By 3.95 of Axler, $\dim V' = \dim V$.
By 2.39 of Axler, $\varphi_1, \dots, \varphi_n$ is a basis of V'. $\blacksquare$

3.99 Definition:

If $T \in \mathcal{L}(V,W)$, then the dual map of T is the linear map $T' \in \mathcal{L}(W',V')$ defined by

$$T'(\varphi) = \varphi \circ T$$

input a linear functional

output a linear functional

Show: $T' \in \mathcal{L}(W',V')$

Let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in W'$ be arbitrary

- Additivity: $T'(\varphi + \psi) = (\varphi + \psi) \circ T$
$= \varphi \circ T + \psi \circ T$
$= T'(\varphi) + T'(\psi)$
- Homogeneity: $T'(\lambda\varphi) = (\lambda\varphi) \circ T$
$= \lambda(\varphi \circ T)$
$= \lambda T'(\varphi)$.

3.100 Example

Define $D: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $Dp = p'$

- Define $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{F})$ by
$$\varphi(p) = p(3).$$

Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ that satisfies

$$(D'(\varphi))(p) = (\varphi \circ D)(p)$$
$$= \varphi(Dp)$$
$$= \varphi(p')$$
$$= p'(3)$$

- Define $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{F})$ by

$$\varphi(p) = \int_0^1 p(x)\,dx$$

Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ given by

$$\begin{aligned}(D'(\varphi))(p) &= (\varphi \circ D)(p)\\ &= \varphi(Dp)\\ &= \varphi(p')\\ &= \int_0^1 p'(x)\,dx\\ &= p(1) - p(0)\end{aligned}$$

3.101 Algebraic Properties of Dual Maps

(a) $(S+T)' = S' + T'$ for all $S, T \in \mathcal{L}(V,W)$

(b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$ and for all $T \in \mathcal{L}(V,W)$

(c) $(ST)' = T'S'$ for all $T \in \mathcal{L}(U,V)$ and for all $S \in \mathcal{L}(V,W)$

Proof: (a): For all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$\begin{aligned}(S+T)'(\varphi) &= \varphi \circ (S+T)\\ &= \varphi \circ S + \varphi \circ T\\ &= S'(\varphi) + T'(\varphi).\end{aligned}$$

(b): For all $\varphi \in \mathcal{L}(V, \mathbb{F})$,

$$\begin{aligned}(\lambda T)'(\varphi) &= \varphi \circ (\lambda T)\\ &= \lambda \varphi \circ T\\ &= \lambda T'(\varphi).\end{aligned}$$

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(c): For all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$(ST)'(\varphi) = \varphi \circ (ST)$$
$$= (\varphi \circ S) \circ T$$
$$= T'(\varphi \circ S)'$$
$$= T'(S'(\varphi))$$
$$= (T'S')(\varphi)$$

So $(ST)'(\varphi) = T'S'$

3.102 Definition:

If U is a subspace of V, then the annihilator of U, denoted U^0, is defined by

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$$

3.103 Example

Let U be the subspace of $P(\mathbb{R})$ consisting of all polynomial multiples of x^2, such as $x^2 p(x)$.

If $\varphi \in \mathcal{L}(P(\mathbb{R}), \mathbb{F})$ is defined by

$$\varphi(p) = p'(0),$$

then $\varphi \in U^0$

$x^2 p(x) \in U$, if $p \in P(\mathbb{R})$

And

$$\varphi(x^2 p(x)) = (x^2 p(x))' \big|_{x=0}$$
$$= (2x p(x) + x^2 p'(x)) \big|_{x=0}$$
$$= 2(0)p(0) + 0^2 p'(0)$$
$$= 0$$

3.104 Example

Let e_1, e_2, e_3, e_4, e_5 denote the standard basis of $\mathbb{R}^5$, and let $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ denote the dual basis of $(\mathbb{R}^5)'$.

Suppose

$$U = \operatorname{span}(e_1, e_2)$$
$$= \{(x_1, x_2, 0, 0, 0) \in \mathbb{R}^5 : x_1, x_2 \in \mathbb{R}\}.$$

Show $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

Solution: Recall from Example 3.97 that

$$\varphi_j(x_1, x_2, x_3, x_4, x_5) = x_j$$

For any $j = 1, 2, 3, 4, 5$.

Suppose we have $\varphi \in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

Then
$$\varphi = c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$$
for some $c_3, c_4, c_5 \in \mathbb{R}$. For all $(x_1, x_2, 0, 0, 0) \in U$,

$$\varphi(x_1, x_2, 0, 0, 0) = (c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(x_1, x_2, 0, 0, 0)$$
$$= c_3\varphi_3(x_1, x_2, 0, 0, 0) + c_4\varphi_4(x_1, x_2, 0, 0, 0) + c_5\varphi_5(x_1, x_2, 0, 0, 0)$$
$$= c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0$$
$$= 0$$

So $\varphi \in U^0$, and so

$$\operatorname{span}(\varphi_3, \varphi_4, \varphi_5) \subset U^0$$

Suppose $\varphi \in U^0$. Since $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ is the dual basis of $(\mathbb{R}^5)'$, we can write uniquely as

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$$

for some $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$

Since $e_1 \in U$, we have

$$\begin{aligned} 0 &= \varphi(e_1) \\ &= (c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(e_1) \\ &= c_1\varphi_1(e_1) + c_2\varphi_2(e_1) + c_3\varphi_3(e_1) + c_4\varphi_4(e_1) + c_5\varphi_5(e_1) \\ &= c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\ &= c_1 \end{aligned}$$

Since $e_2 \in U$,

$$\begin{aligned} 0 &= \varphi(e_2) \\ &= (c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(e_2) \\ &= c_1\varphi_1(e_2) + c_2\varphi_2(e_2) + c_3\varphi_3(e_2) + c_4\varphi_4(e_2) + c_5\varphi_5(e_2) \\ &= c_1 \cdot 0 + c_2 \cdot 1 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\ &= c_2 \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi &= c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \\ &= 0\varphi_1 + 0\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \\ &= c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \\ &\in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5) \end{aligned}$$

So $U^0 \subset \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

Therefore, $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

3.105 The annihilator is a subspace

Suppose U is a subspace of V.
Then U^0 is a subspace of V'.
$(V' = \mathcal{L}(V, \mathbb{F}))$

Proof:

- Additive identity: Since $0(u) = 0$ for all $u \in U$, we have $0 \in U^0$
- Closed under addition: Suppose $\varphi, \psi \in U^0$. For all $u \in U$, we have

$$(\varphi + \psi)(u) = \varphi(u) + \psi(u)$$
$$= 0 + 0$$
$$= 0$$

So $\varphi + \psi \in U^0$.

- Closed under scalar multiplication:
Suppose $\lambda \in \mathbb{F}$ and $\varphi \in U^0$. For all $u \in U$, we have

$$(\lambda\varphi)(u) = \lambda\varphi(u)$$
$$= \lambda \cdot 0$$
$$= 0$$

So $\lambda\varphi \in U^0$

$\blacksquare$

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3.111 Def:

The transpose of a matrix A, denoted A^t, is the matrix obtained from A by interchanging the rows and columns.

More specifically, if

$$A = \begin{pmatrix} A_{11} & A_{1m} \\ A_{m1} & A_{mn} \end{pmatrix} \text{ then } A^t = \begin{pmatrix} A_{11} & A_{m1} \\ A_{1n} & A_{mn} \end{pmatrix}$$

3.112 Ex

If $A = \begin{pmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{pmatrix}$, then $A^t = \begin{pmatrix} 5 & 3 & -4 \\ -7 & 8 & 2 \end{pmatrix}$

3.113 The transpose of the product of matrices

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then

$$(AC)^t = C^t A^t$$

Proof: Suppose we have
$k=1,\dots,p$ and $j=1,\dots,m$

Then $((AC)^t)_{k,j} = (AC)_{j,k}$

$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

$$= \sum_{r=1}^{n} (A^t)_{r,j} (C^t)_{k,r}$$

$$= \sum_{r=1}^{n} (C^t)_{k,r} (A^t)_{r,j}$$

$$= (C^t A^t)_{k,j}$$

Therefore, we have

$$(AC)^t = C^t A^t \quad \blacksquare$$

3.114 The matrix of T' is the transpose of the matrix of T

Suppose $T \in \mathcal{L}(V,W)$. Then

$$M(T') = (M(T))^t$$

Proof: let $A = M(T)$ and $C = M(T')$

Suppose we have $j=1,\dots,m$ and $k=1,\dots,n$.

By definition 3.32 of Axler, we have

$$Tv_k = \sum_{r=1}^{m} A_{r,k} w_r$$

and $T'(\psi_j) = \sum_{r=1}^{n} C_{r,j} \varphi_r$

So we have

$$(\psi_j \circ T)(v_k) = (T'(\varphi_j))(v_k)$$
$$= \left(\sum_{r=1}^{n} C_{r,j} \varphi_r \right)(v_k)$$
$$= (C_{1,j}\varphi_1 + \dots + C_{n,j}\varphi_n)(v_k)$$
$$= C_{1,j}\varphi_1(v_k) + \dots + C_{n,j}\varphi_n(v_k)$$
$$= C_{1,j}\overset{0}{\cancel{\varphi_1(v_k)}} + \dots + C_{k,j}\overset{1}{\cancel{\varphi_k(v_k)}} + \dots + C_{n,j}\overset{0}{\cancel{\varphi_n(v_k)}}$$
$$= C_{k,j}\varphi_k(v_k)$$

and

$$(\psi_j \circ T)(v_k) = \varphi_j(Tv_k)$$
$$= \psi_j\left(\sum_{r=1}^{n} A_{r,k} w_r \right)$$
$$= \psi_j(A_{1,k}w_1 + \dots + A_{n,k}w_n)$$
$$= A_{1,k}\psi_j(w_1) + \dots + A_{n,k}\psi_j(w_n)$$
$$= A_{1,k}\overset{0}{\cancel{\psi_j(w_1)}} + \dots + A_{j,k}\overset{1}{\cancel{\psi_j(w_j)}} + \dots + A_{n,k}\overset{0}{\cancel{\psi_j(w_n)}}$$
$$= A_{1,k}\cdot 0 + \dots + A_{j,k}\cdot 1 + \dots + A_{n,k}\cdot 0$$
$$= A_{j,k}$$

Therefore, we conclude

$$C_{k,j} = A_{j,k}$$

So $C = A^t$,

and so $M(T') = C$

$= A^t$

$= (M(T))^t$ $\blacksquare$

Back to Annihilator

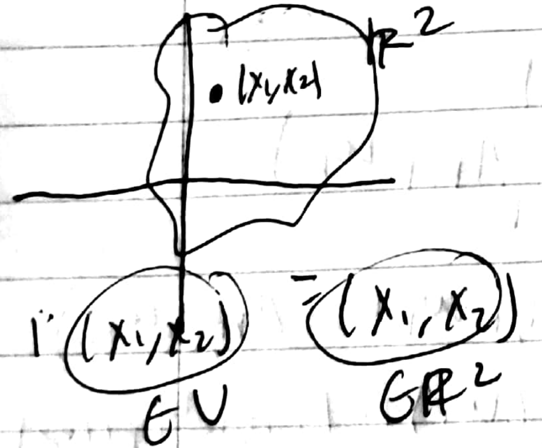
(3.106 - 3.110)

3.106 Dimension of the annihilator

Suppose V is a finite-dimensional vector space and U is a subspace of V. Then

$$\dim U + \dim U^0 = \dim V$$

Proof: Let $i \in \mathcal{L}(U, V)$ be the inclusion map defined by $i(u) = u$ for all $u \in U$.



First, we will show that i' is linear. Let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in V'$

- Additivity: For all $u \in U$, we have

$$(i'(\varphi + \psi))(u) = ((\varphi + \psi) \circ i)(u)$$

$$= (\varphi + \psi)(i(u))$$
$$= (\varphi + \psi)(u)$$
$$= \varphi(u) + \psi(u)$$
$$= \varphi(i(u)) + \psi(i(u))$$
$$= (\varphi \circ i)(u) + (\psi \circ i)(u)$$
$$= (i'(\varphi))(u) + (i'(\psi))(u)$$
$$= (i'(\varphi) + i'(\psi))(u)$$

Therefore,

$$i'(\varphi + \psi) = i'(\varphi) + i'(\psi)$$

◦ Homogeneity: For all $u \in U$, we have

$$(i'(\lambda\varphi))(u) = ((\lambda\varphi) \circ i)(u)$$
$$= (\lambda\varphi)(i(u))$$
$$= (\lambda\varphi)(u)$$
$$= \lambda\varphi(u)$$
$$= \lambda\varphi(i(u))$$
$$= \lambda((\varphi \circ i)(u))$$
$$= (\lambda(i'(\varphi)))(u)$$

Therefore, $i'(\lambda\varphi) = \lambda i'(\varphi)$

Next, we will show null $i' = U^0$.
We have

$$\text{null } i' = \{\varphi \in V' : i'(\varphi) = 0\}$$
$$= \{\varphi \in V' : \varphi \circ i = 0\}$$
$$= \{\varphi \in V' : (\varphi \circ i)(u) = 0 \text{ for all } u \in U\}$$
$$= \{\varphi \in V' : \varphi(i(u)) = 0 \text{ for all } u \in U\}$$
$$= \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$$
$$= U^0$$

By the Fund. Thm of Linear Maps [3.22 of Axler], we have

$$\dim V = \dim V' \quad \text{by 3.95 of Axler}$$
$$= \dim \text{range } i' + \dim \text{null } i' \quad \text{Fund. Thm of Linear Maps (3.22)}$$
$$= \dim \text{range } i' + \dim U^0$$
$$= \dim U' + \dim U^0$$
$$= \dim U + \dim U^0$$

once we show range $i' = U'$

Show range $i' = U'$

Suppose we have $\varphi \in U'$. By exercise 3.A.11 of Axler, we can extend to a linear functional ψ on V. And by definition of i' we have $i'(\psi) = \varphi$. So $\varphi \in$ range i', and so we have $U' \subset$ range i'. But 3.19 of Axler says that range i' is a subspace of U'. Therefore, range $i' = U'$, as desired $\blacksquare$

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3.107 The nullspace of T'

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V,W)$. Then

(a) $\operatorname{null} T' = (\operatorname{range} T)^0$

(b) $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$.

Proof: (a) Suppose $\varphi \in \operatorname{null} T'$. Then we have $T'(\varphi) = 0$. So for all $v \in V$, we have

$$\begin{aligned} 0 &= 0(v) \\ &= (T'(\varphi))(v) \\ &= (\varphi \circ T)(v) \\ &= \varphi(Tv) \end{aligned}$$

So $\varphi \in (\operatorname{range} T)^0$, and so we have

$$\operatorname{null} T' \subset (\operatorname{range} T)^0$$

Suppose $\varphi \in (\operatorname{range} T)^0$. Then $\varphi(Tv) = 0$ for all $v \in V$. So we have

$$\begin{aligned} (T'(\varphi))(v) &= (\varphi \circ T)(v) \\ &= \varphi(Tv) \\ &= \varphi(0) \\ &= 0 \end{aligned}$$

and so $\varphi \in \operatorname{null} T'$. So $(\operatorname{range} T)^0 \subset \operatorname{null} T'$.

Therefore, $(\operatorname{range} T)^0 = \operatorname{null} T'$.

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(b) ∵ we have

$$\dim \operatorname{null} T' = \dim (\operatorname{range} T)^0 \text{ by part (a)}$$
$$= \dim W - \dim \operatorname{range} T \text{ by 3.106 of Axler}$$
$$= \dim W - (\dim V - \dim \operatorname{null} T)$$
by Fund. Thm of Lin Maps (3.22)
$$= \dim \operatorname{null} T + \dim W - \dim V$$

∎

3.108 T surjective is equivalent to T' injective

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then
T is surjective if and only if T' is injective

Proof: $\iff$ = "if and only if"

$T \in \mathcal{L}(V, W)$ is surjective $\iff$ range $T = W$
$\iff (\operatorname{range} T)^0 = \{0\}$
$\iff \operatorname{null} T' = \{0\}$ by 3.107(a) of Axler
$\iff T'$ is injective

Prove $(\operatorname{range} T)^0 = \{0\}$

we have $\dim (\operatorname{range} T)^0 = \dim W - \dim \operatorname{range} T$ by 3.106 of Axler
$= \dim W - \dim W$
$= 0$
$= \dim \{0\}$

if and only if $(\text{range } T)^0 = \{0\}$, by Exercise 2.C.1 of Axler. $(\Leftarrow ?)$

3.109 The range of T'

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then

(a) $\dim \text{range } T' = \dim \text{range } T$,

(b) $\text{range } T' = (\text{null } T)^0$.

Prove: (a): We have

$$\begin{aligned} \dim \text{range } T' &= \dim W' - \dim \text{null } T' && \text{by Fund. Thm of Lin Maps (3.22)} \\ &= \dim W - \dim \text{null } T' && \text{by 3.95 of Axler} \\ &= \dim W - \dim (\text{range } T)^0 && \text{by 3.107(a)} \\ &= \dim \text{range } T && \text{by 3.106} \end{aligned}$$

(b): Suppose $\varphi \in \text{range } T'$. Then there exists $\psi \in W'$ that satisfies $\varphi = T'(\psi)$. For all $v \in \text{null } T$, we have

$$\begin{aligned} \varphi(v) &= (T'(\psi))(v) \\ &= (\psi \circ T)(v) \\ &= \psi(Tv) \\ &= \psi(0) && \text{since } v \in \text{null } T \\ &= 0. \end{aligned}$$

So $\varphi \in (\text{null } T)^0$. Therefore, $\text{range } T' \subset (\text{null } T)^0$.

Show $\dim \text{range } T' = \dim (\text{null } T)^0$

we have

$$\dim \text{range } T' = \dim \text{range } T \text{ by } 3.109(a)$$
$$= \dim V - \dim \text{null } T \text{ by Fund. Thm of Lin Maps (3.22)}$$
$$= \dim (\text{null } T)^0 \text{ by } 3.106$$

Therefore, we have in fact range $T' = (\text{null } T)^0$.

3.110 T injective is equivalent to T' surjective

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V,W)$. Then T is injective if and only if T' is surjective.

Proof: $T \in \mathcal{L}(V,W)$ is injective $\iff$ null $T = \{0\}$ by 3.16 of Axler

$\iff (\text{null } T)^0 = V'$

$\iff$ range $T' = V'$ by 3.109(b) of Axler

$\iff T'$ is surjective

Prove $(\text{null } T)^0 = V'$

we have

$$\dim (\text{null } T)^0 = \dim V - \dim \text{null } T \text{ by } 3.106 \text{ of Axler}$$
$$= \dim V - \dim \{0\}$$
$$= \dim V - 0$$
$$= \dim V$$
$$= \dim V' \text{ by } 3.95 \text{ of Axler}$$

if and only if $(\text{null } T)^0 = V'$, by Exercise 2.C.1 of Axler